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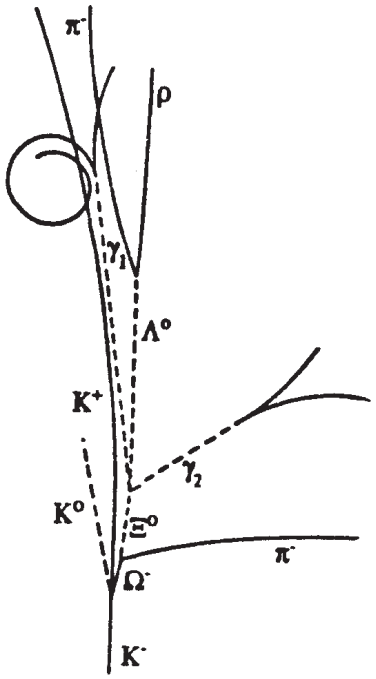

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DEFINING ELECTRIC CHARGE CURRENT RESISTANCE VOLTAGE TEMPERATURE MAGNETIC PRESSURE

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Abstract

Here is a deconstruction of the ideal gas equation. This has implications in our understanding of the terms which describe temperature, pressure, electromagnetism and mass. Understanding of these terms, are arrived at from experiment. Electric charge squared is a constant which equals a particle mass times distance travelled. Electric current is now defined as the momentum of one coulomb of particles such as photons, electrons, protons, ions, atoms. Electric resistance is defined as velocity per elementary charge or electric current per mass of one particle as described in the electric current definition. The symbol V in electron volts eV stands for acceleration a. Temperature is redefined as electric acceleration ea. Magnetic pressure is defined as the square of a magnetic field B.

Keywords: 186-ether, Graviton, Aitheron, Quantum, Hall, Boltzmann constant, Proton, Electron, Electron Volt, Current, Wavelength, Frequency, Charge, Mass, Molar Volume, Magnetic Pressure, Temperature, Ideal Gas Equation, Avogadro, Klaus Von Klitzing, Josephson, Charles, Kelvin, Boyle, Pascal, Gay-Lussac, Feynman, Pendulum, Wien, Planck

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1 Introduction

Blaise Pascal is known for his work on pressure. Lord Kelvin devised the absolute temperature scale. Avogadro, Boyle, Kelvin, Charles, Gay-Lussac have contributed to the Ideal Gas Equation laws. Feynman raised an important problem in Physics in the hazy understanding of units of energy especially the unit of electron volts. He apologizes for the units of energy with respect to eV. Another difficulty brushed aside is in the physical meaning of the terms electric current, voltage, resistance and charge. The world carries on as if this is elementary stuff the level of high school students and worked out in the context of far bigger advances in Physics. This is far from the truth as the reader will soon find out.

2. Literature review

In the table below I have deconstructed everyday terms by electricians, chemists, biologists, engineers and doctors. Through an example of a cardiac heart jumpstart device, I place my arguments with empirical data. You will notice that there is no evidence of electron action in this cardiac revival device and that PHOTONS are involved in CAPACITANCE. https://www.naturalphilosophy.org/pdf/abstracts/abstracts_7287.pdf

On June 20th 2024 Ethan Sagel said – Ohm's Law V = IR cannot be derived.

My response – Without a doubt Newton's F = ma is the same equation as Ohm's Law V = IR.

Acceleration, a $= \frac{eV}{e} = I$ x R

Resistance, R $= \frac{c}{e} = \frac{I}{m}$

Acceleration, a $= I$ x R $= I$ x $\frac{I}{m}$

Force, F = ma $= I$ x I

This is a unifying theory of different symbols for force phenomena.

Consider a cardiac defibrillator device that discharges 0.17C, 15.0 amps, 2.5×10^3 volts to the heart in 11.333 milliseconds from a capacitor at 68×10^{-6} Farads

Coulombs	0.17 C		1
Amperes	15.0 A		2
Volts	2.5×10^3 V		3
Time period	0.0113333 s		4
Capacitance	68×10^{-6} F		5
Resistance	Ohms=V/I=166.666 Ω		6
Inductance	$L = \Omega \times$ s $L = 1.888883333H$	$L = \Omega \times t$ $L = 166.666 \times 0.0113333$ $L = 1.888883333H$	7
String length D of one photon	$D = L \times eD$ $= 3.026333446 \times 10^{-19}m$	L is the string length of 6.24×10^{18} photons $m^* = \frac{(1.602176634 \times 10^{-19})^2}{3.026333446 \times 10^{-19}}$ $m^* = 8.48211041 \times 10^{-20}kg$	8
String length R of # photons	# = Q/e = 0.17 /e # = $1.06105661 \times 10^{18}$	$R = D \times \# = 0.321111$m	9
Acceleration is Voltage	a = R/t²	a = 0.321111 / (0.0113333)² a = 2.5×10^3 m/s²= Volt	10
$Q^2 = m \times r$	$0.17^2 = m \times 0.32111$	$m = 0.09kg$	11
$e^2 = m^{\cdot} \times r^{\cdot}$	m^* $= 8.016428171 \times 10^{-27}kg$	$\Phi = \frac{h}{2e}$ $\Phi = 2.0678337171e - 15Wb$ $\Phi = 15.0 \times m^* \times 10^{-7} \times 137.036Wb$ r* =3.2021363864885e-12m	12
$E = m^{\cdot\cdot} \times c^2$	$m^{\cdot\cdot} = 7.5763127 \times 10^{-34}kg$	$E = Q \times V \times e$ $E = 0.17 \times 2.5 \times 10^3 \times eJ$	13
$q^2 = m^{\cdot} \times D$	$m^{\cdot}{}_{kg}$ $= 8.48211041 \times 10^{-20}kg$	$m^{\cdot\cdot}{}_{kg} = \frac{E}{c^2} = \frac{6.809250253 \times 10^{-17}}{(2.99792458 \times 10^8)^2}$ $= 7.576312676 \times 10^{-34}kg$	14
Resistance $\Omega = \frac{l}{m^{\cdot}} = \frac{c}{e}$	$m^* = 8.016428989$e-27Kg $\Omega = \frac{15.0}{m^{\cdot}} = \frac{c}{e}$		15
Resistance $\Omega = \frac{vel}{Q}$	$\Omega = 166.666 = \frac{\lambda}{t_1 \times t_2} \times \frac{1}{l}$ $= \frac{Volt}{I}$ $h = m^{\cdot\cdot} \times c \times \lambda$ $= m^{\cdot} \times v \times r^{\cdot\cdot}$	$v = \frac{0.321109222}{0.0113333} = 28.3333m/s$	16
Pendulum	$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$	$\frac{g}{l} = \frac{Volt}{R} = \frac{1}{LC} = 7{,}785.49406745352Hz^2$ g is acceleration, a of an ether toroid Ex in (10)(12)	17
Induction	$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$	$\frac{2.5 \times 10^3}{0.32111} = \frac{1}{1.888883333 \times 68 \times 10^{-6}}$	18

3. Method

Employing ratio and proportion where dimensional homogeneity is maintained on both sides of an empirical equation, I present these empirical equations utilizing algebraic computation.

4. Data Processing and Analysis

Angular Momentum of Ether the cloak of Boltzmann constant

$Q^2 = e^2$ squared $\times [8.610225 \times 10^{-5}]^2$ {Q=Boltzmann constant} (1)

PV = Energy, E = kT (2)

$E = (2\pi \times 10^{-7} \times 137.036 \times e) \times (e \times a) = k \times T$ (3)

$\left(\frac{I^2}{R_1 \times R_2}\right) \times (R_1 \times R_2 \times R_3) = (2\pi \times 10^{-7} \times 137.036 \times e) \times (e \times a)$ (4)

$\frac{F}{A} \times V = 2\pi \times 10^{-7} \times 137.036 \times E$ (5)

Atmospheric pressure is $\frac{F}{A}$ or $\frac{I^2}{R_1 \times R_2}$ (6)

Magnetic Pressure P = Square of magnetic field B = Gas pressure = Current squared / Area

Boltzmann $k \equiv Q = 2\pi \times 10^{-7} \times 137.036 \times e$

$PV = Q \times T$ where $T = eV = ea$ (7)

Repurposing the Ideal Gas Equation: $mc^2 = Q \times T$ (8)

The role of temperature has been underestimated by concluding that average kinetic energy is a final definition for temperature. In-fact over several decades various temperature scales were invented based on arbitrary units of measurements leading to conversion tables.

Temperature is defined as being proportional to the *average* kinetic energy of a body; "average" expresses the reality that it is unfeasible to measure different areas of a body possessing different energies.

The next experiment will lead to a new paradigm in defining Kelvin T as electron volts or more accurately elementary charge e times acceleration of a photon.

Temperature T = eV = ea

Temperature is in absolute fact electron volts, where electron volts is a misnomer. Electric acceleration is a suggested right descriptor.

In the experiment that follows I have added one step to Wien's Law: Empirical evidence as to why two different frequencies (wavelength) represent the same associated temperature.

It is a simple proof that demonstrates empirically the relationship between two black body radiations laws namely, Wien's law and Planck's derivation law.

Electron volt eV = Temperature, T	Kelvin, K = Temperature, T_2
• Apply Wien's law, $2.9 \times 10^{-3} = \lambda_1 \times T$ $2.9 \times 10^{-3} = \lambda_1 \times eV$ $\lambda_1 \times (2\pi \times 10^{-7} \times 137.036) = \lambda_2$ Example: eV = 511000 and the Compton wavelength for an electron of mass m	$2.9 \times 10^{-3} = \lambda_2 \times T_2$ • Apply Planck's Law, $0.0144 = \lambda_3 \times T_2$ • The photon mass m at wavelength obeys de Broglie's equation $m \times c \times \lambda_3 = h$

A rotational factor of $2\pi \times 10^{-7} \times 137.036$ radians explain why the same Kelvin temperature of a body radiates two differently measured frequencies.

Electric Acceleration: $V = \frac{eV}{e} = acceleration\ a$ A paradigm shift.

The V symbol in eV is acceleration – accelerating voltage.
It is not EMF
It is not PD

It is not Energy

Idiocy of energy units – It's too bad, but I have already apologized, and there is nothing else I can do… - Richard P. Feynman (1961)

Consider 511 keV for an electron, (10) and (12) have the same magnitude by 24 decimal places.

$$\frac{eV}{e} = V \tag{9}$$

$$\frac{511000}{1.60217653\times10^{-19}} = 3.1894 \times 10^{24} \text{ Volts} \tag{10}$$

$$\text{Acceleration} = \frac{c^2}{r} \tag{11}$$

$$\frac{(2.998\times10^{8})^2}{2.817940325\times10^{-8}} = 3.1894\times10^{24}\ m/s^2 \tag{12}$$

$$\text{Voltage, } V = \frac{c^2}{r} \tag{13}$$

The term eV is called electron volt by convention. This is a misnomer. The e stands for elementary charge and not only that of an electron. The symbol V is acceleration. The term eV = Joules per coulomb. eVe is energy in joules.

eV is not energy as is loosely stated in text. Rather eV is the energy of 6.24e18 particles. And eVe the energy of one particle. *It is imperative to call the V in eV accelerating voltage*.

a = **f** × VRK $m \times VRK$ VRK = 25812.807 m/s This acceleration a is that of the *Ether toroid* producing a force on 186 Ether of 1.2102737×10e44N which is at speed of light squared. This is the solution for superconductivity at Tc. REF [4] SIPS CRETE 2024

The thought process continues in seemingly unrelated experiments like the pendulum.

Unification of Gravity with Electromagnetism (GEM)

$$T = 2\pi\sqrt{\frac{ml}{mg}} \tag{14}$$

$$T = 2\pi\sqrt{\frac{q^2}{I^2}} \quad (15)$$

$$T = 2\pi\frac{q}{I} \quad (16)$$

$$q = I \times T/2\pi \quad (17)$$

In 1961 Feynman uttered the apologies for the units of energy. Now, I have solved the problem at the swing of a pendulum. Prof. Calculus in Tintin comics has created the wonders associated with the simple pendulum deep in my mind.

Defining mass m: 737 Aitheron × f = m (18)

Defining velocity VRK: $h = m \times VRK \times r = e^2 \times VRK$ $VRK = \frac{h}{e^2}$ (19)

$I = \frac{mc}{e} = \frac{Mv}{e}$ Motion of electrons limits our understanding of electric current (20)

Mass m or mass M can represent a photon, an atom, an ion, ether mass

$$R\ ohms = \frac{I}{m} = \frac{c}{e} = 1.871157347061897046739729197 6735 \times 10^{27} ohms \quad (21)$$

Variable resistance $R\ ohms = \frac{v}{e}$

$\frac{eV}{e} = a$ $\frac{mc}{e} \times \frac{v}{e} = IR$ For one particle [Ref. 3] (22)

5. Results and Discussions

Fernandes Research Outcomes and Equations:

737 Aitheron: $7.372497321 \times 10^{-51}$ Kg

Graviton: $7.372497321 \times 10^{-58}$ Kg

Ether-Graviton

$$1.859222909 \times 10^{-9} \times V_{Rk} = 7.372497321 \times 10^{-58} \times \frac{c^2}{1.380668031 \times 10^{-36}} \times 1.0 \quad (23)$$

The energy of 737 Aitheron equals Planck's constant times one second.

$$7.372497321 \times 10^{-51}\ \text{Kg} \times c^2 = h \times 1.0\ \text{s} \tag{24}$$

$$1.859222909 \times 10^{-9} \times V_{VRK} = 7.372497321 \times 10^{-51} \times \frac{c^2}{1.380668031 \times 10^{-29}} \times 1.0 \tag{25}$$

So, the imposed condition is 1 second for the wave-maker measurement.

737 Aitheron Frequency: A defining moment for understanding the meaning of mass

Superconducting Velocity VRK = 25812.807 m/s

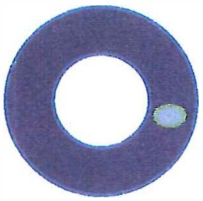
The Quantum Hall Resistance, RK = 25,812.807 Ω/C units of Ohms / 1Coulomb

Faster Than Light FTL = $3.481818846 \times 10^{12}$ m/s This is the temperature photon velocity

Speed of Light,
c = Aitheron frequency × 2π × Radius of an Ether Toroid × 137.036

Speed of Light Squared, c^2 = VRK × FTL = $25812.807 \times 3.481818846 \times 10^{12}$

186 Ether mass [grey] × acceleration c^2/r = Ether toroid mass[blue-black] × a= 1.21e44 N



Wavelength Lambda = 2Pi × R × 137.036

Magnetic Flux Quantum Phi = BA A = Pi × R × r × 137.036

Mass is a measure of the frequency of a 737 Aitheron:

7.372497321e-51 Kg.s × f = Mass

6. Conclusion

A deeper understanding of primary and secondary units in Physics sets the scientific world on a new course of discovery. Ad hoc trial and error experimental design repeats stem from physical constants such as the Boltzmann constant being wrongly thought of. Here I am hitting at statistical outcomes versus algebraic computation of empirical data. Statistical outcomes are approximations and missed assumptions and variables.

Terms used in Physics have now been rewritten based on experiments. A paradigm shift in several terms is presented in the table below. The cardiac defibrillator, simple pendulum and BB radiation equations all point to some phenomenon being measured differently.

Paradigm shifts always lead to technological skips. The table below depicts paradigm shifts based on empirical evidence.

Term	Standard Physics Definitions	Fernandes Definitions
Voltage	EMF PD eV V	Acceleration
Current	Motion of Electrons	Momentum of 1 Coulomb Photons, Ions, Atoms, Electrons, 186-Ether
Resistance	Resists the flow of electrons	Velocity of 1 Coulomb Photons, Ions, Atoms, Electrons, 186-Ether
Wavelength	Distance crest to crest	Circumference of a circle × 137.036

Frequency of a wave	Cycles per second – Unit of Hz	Cycles per second – unit of Hz that of an Aitheron
Charge	Gives direction + or -	Mass/C × Wavelength
Charge squared	-	Mass × Radius of a Torus
Acceleration	Rate of change of speed	Depends on temperature photon T ≈ eV = ea
Newton's Law	F = ma	Ohm's Law V = IR
Gravity, g	Acceleration due to gravity	Acceleration of Ether
Temperature	Average KE	Electron volt or ea
Energy	Mass converts to energy at lightspeed	Energy is a measurement with no conversion at lightspeed
Fringe	Dispersion of light on a screen	Ether fringe
Hand of God Alpha	A dimensionless constant	A toroidal pulsation factor
Ideal Gas Equation	PV = nKT	$B^2 \times R1 \times R2 \times R3 = n$ Q × ea
Volume of one mole gas	Avogadro's number at fixed T and P	Three lengths R1×R2×R3 of photons at fixed T and P

TECHNOLOGICAL SIGNIFICANCE – In Medicine

Bioengineering – Catalysis is about lowering the activation energy which is now understood as eV of a temperature photon. Enzymes cut and paste biomolecules in drug making strategies. Photonics can fine tune this process.

PCR – This process of gene amplification is temperature dependent. A further understanding of gene amplification via temperature photonics needs to be explored.

Inflammation – Temperature and inflammation are related phenomena. Thermometers for fever measurement are in use in monitoring patients in hospitals with infections.

Biomedical Imaging – Superconductors in imaging equipment and logic circuits in chips in electronic devices, wearables can be efficiently designed with right equations of electrical terms.

NIST – CODATA

Quantized Hall Resistance, RK = 25,812.807 Ω

Fine Structure Constant alpha = 0.00729735257 $\frac{1}{\alpha}$ = 137.0359991 rounded off to 137.036

Speed of Light, c = 2.99792458×10^{8} m/s

Planck constant, h = $6.62607015 \times 10^{-34}$ J.s

Elementary charge, e = $1.602176634 \times 10^{-19}$ C

Josephson constant, Kj = $4.8359784841698 \times 10^{14}$ Hz/V

Magnetic Flux Quantum – Phi,$\emptyset = \frac{h}{2e} = 2.067833848 \times 10^{-15}$Wb

Gravitational Constant G = $6.6742 \times 10^{-11} \frac{m}{Kg} \times \frac{m^2}{s^2}$

Standard Physics Equations

Electric Field: $E = \frac{F}{e}$

Magnetic Field: $B = \frac{F}{ev}$

Force = $eE = \frac{eV}{d} = Bev$

Wavelength: $\lambda = \frac{h}{mc}$

Hall Voltage: $V = I \times 25812.807$

Josephson Equation: $f_I = \left(\frac{2e}{h}\right) V$ $\frac{2e}{h} = 4.836 \times 10^{14}$

References

[1] Francis V. Fernandes, The Hand of God Alpha Represents Radial Pulsation Limits of Fernandes-186 Ether to the Planck Ether Length, Sips23_70_407FS
[2] Francis V. Fernandes, 144 GeV Atlas Bump Reveals the Fernandes 186-photon Mass, Sips23_70_408FS
[3] Francis V. Fernandes, the Higg's Boson Does Not Give Mass, the Fernandes 737-Aitheron Frequency does, Sips23_70_409FS
[4] Francis V. Fernandes, Electrodynamic Treatment of Infectious Diseases - The Realm of Biophysics, Crete SIPS 2024

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THE ASTONISHING CONFLICT OF THE CONSTANCY OF THE ONE-WAY SPEED OF LIGHT WITHIN RELATIVITY

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Abstract

We show that if we assume the existence of a frame in vacuum where the one-way speed of light is c (c is the measured value of the two-way speed of light in vacuum) than for another frame moving with velocity v_1 in relation to that frame the speed of light in this another frame cannot be c. What is c in this another frame is the one-way Einstein speed of light. Although this has been known and affirmed several years ago, only recently we have an increase of publications stating the variability of the one-way speed of light. Therefore, since the constancy of the speed of light is sometimes considered an unquestionable postulate, we are assisting to an astonishing conflict between the "standard formulation of Relativity" and "Relativity with variable light speed". However, this conundrum can be easily resolved with the clarification of the conceptualization of simultaneity and synchronization.

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Introduction

Recently one more interesting article questioned the accuracy of the postulate of the constancy of the speed of light has been published [1].
If the constancy of the speed of light is not true, the postulate must be reconsidered.
For an unwary reader we advise that the one-way speed of light in one frame can be *c/2* and infinite for the opposed direction and simultaneously the Einstein speed of light is *c*.
In 2002 it has been discovered [2-4] that the one-way speed of light is $\frac{c}{1+\frac{v_1}{c}}$ and $\frac{c}{1-\frac{v_1}{c}}$ for opposed directions and with Lorentzian clocks the Einstein speed of light continued to be c - we discover the intrinsic desynchronization that standard formulation of course cannot be aware [5, 6, 7-111]. Even today we assist to a permanent discussion of the twin paradox conundrum and the Sagnac effect conundrum [110, 111].

I. The Speed of Light in a Frame moving in relation to EF (Einstein Frame) the frame where the one-way speed of light is isotropic with the limit value *c*.

Standard formulation affirm that the speed of light is c whatever the frame considered. Therefore, we can consider one frame where the speed of light is c and consider another frame moving with velocity v_1 in relation to that frame and consider also that the two-way speed of light in this frame is c since in vacuum it is what we measure as also standard formulation assume.

If the constancy of the speed of light is not true, the postulate must be reconsidered. For an unwary reader we insist, the one-way speed of light can be *c/2* and infinite for opposed directions. It is not only the problem of experimental difficulty to measure.

With these premises we obtain the one-way speed of light as a result of Lorentz-FitzGerald contraction and Larmor Time dilation [7, 17] in relation to *EF*. Note that we are not assuming that we can consider every frame as *EF* because we can't assume that as standard formulation does – the equivalence of the frames. This error is the origin of the astonishing conflict as Zbigniew Oziewicz refer [5, 6].

Indeed, consider frames S and S'. S is the *EF* and S' is a frame moving in a standard configuration in relation to S with velocity v_1. This standard

configuration is S' moving through x of S. O′ the origin of S' coincide with O the origin of S and O′ moves through the x axis of S with v_1. When coincide with O light is emitted from O′ through x' and light is also emitted from O through x.

The one-way light-velocity is source-free in one and only one reference system. This reference system is said to be the Einstein lost frame (Abreu Axiom as Zbigniew Oziewicz define it [5]).

The light emitted through x' pass by x' with

$$x' = c't', \qquad (1)$$

c' is the speed of light and $t'$is the synchronized time of a clock located at x'.

This light emitted from O′ through S moves with speed c because S is the EF and the speed of light is independent of the speed of the source O′ from S' moving with speed v_1 [5]. The coordinate x' move through x with eq.

$$x = v_1 t + x'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (2)$$

where t is the synchronized time of S, and $x'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}$ is the Lorentz-FitzGerald contraction.

Therefore, when light arrives at x we have,

$$ct = v_1 t + x'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (3)$$

From (1) and (3)

$$ct = v_1 t + c't'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (4)$$

$$(c - v_1)t = c't'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (5)$$

Since

$$t' = t\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (6)$$

we obtain

$$(c - v_1)t = c't\left(1 - \frac{{v_1}^2}{c^2}\right) \qquad (7)$$

$$c' = \frac{c}{1 + \frac{v_1}{c}} \qquad (8)$$

Therefore, only for $v_1 = 0$ $c' = c$. For the opposed direction

$$c' = \frac{c}{1 - \frac{v_1}{c}} \qquad (9)$$

The two-way speed is c.

Therefore, we have a simple explanation for the astonishing acceptance of the obvious mistake in relation to the value of the speed of light. What is c is the Einstein speed of light as we are going to show in this context.

The Einstein speed of light is c of course because it is not the speed of light. It is the distance of the trip divided by the difference of the Lorentzian times of the clocks at arrival and departure. Since this clocks are desynchronized by the intrinsic desynchronization introduced by us [2-4, 5, 10] the value of the two way speed of light is obtained for the Einstein speed of light. Indeed, from (8) we obtain the intrinsic desynchronization, the following eq. (12)

$$\frac{x'}{c'} = \frac{x'}{c}\left(1 + \frac{v_1}{c}\right) = \frac{x'}{c} + \frac{x'v_1}{c^2} \qquad (10)$$

$$t' = t'_L + \frac{v_1}{c^2}x' \qquad (11)$$

$$t'_L = t' - \frac{v_1}{c^2}x' \qquad (12)$$

If light is emitted from O′ to x' the clock at x' mark $t_L^{'} = \frac{x'}{c}$ and is initiated when light arrives. This is the synchronization of Einstein (a desynchronization). Since a true synchronization correspond to $t' = \frac{x'}{c'}$ the value of the desynchronization is from (12) $\frac{v_1}{c^2}x'$.The Lorentzian clock mark $t'_L = \frac{x'}{c}$ and t'_L is the Lorentzian time.

II. Lorentz Transformation from IST Transformation using intrinsic desynchronization.

From (2) and (6), Lorentz-FitzGerald contraction and Time dilation we have the IST Transformation

$$x = v_1 t + x'\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (13)$$

or

$$x' = \frac{x - v_1 t}{\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}} \qquad (14)$$

$$t' = t\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} \qquad (15)$$

From (6) and (14) with

$$t_L^{'} = t' - \frac{v_1}{c^2}x' \qquad (16)$$

we obtain

$$t_L^{'} = t' - \frac{v_1}{c^2}x' = t\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)} - \frac{v_1}{c^2}\frac{x - v_1 t}{\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}} \qquad (17)$$

$$t_L^{'} = t' - \frac{v_1}{c^2}x' = \frac{t - \frac{v_1}{c^2}x}{\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}} \qquad (18)$$

$$x' = \frac{x - v_1 t}{\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}} \qquad (19)$$

and

$$t_L' = \frac{t - \frac{v_1}{c^2} x}{\sqrt{\left(1 - \frac{{v_1}^2}{c^2}\right)}} \qquad (20)$$

Therefore we obtain for S'' moving with v_2

$$x'' = \frac{x - v_2 t}{\sqrt{\left(1 - \frac{{v_2}^2}{c^2}\right)}} \qquad (21)$$

$$t_L'' = \frac{t - \frac{v_2}{c^2} x}{\sqrt{\left(1 - \frac{{v_2}^2}{c^2}\right)}} \qquad (22)$$

And from ((19)-(22)) we obtain

$$x'' = \frac{x' - v_E' t_L'}{\sqrt{\left(1 - \frac{{v_E'}^2}{c^2}\right)}} \qquad (23)$$

and

$$t_L'' = \frac{t_L' - \frac{v_E'}{c^2} x'}{\sqrt{\left(1 - \frac{{v_E'}^2}{c^2}\right)}} \qquad (24)$$

with Einstein velocity given by

$$V_E' = \frac{v_2 - v_1}{1 - \frac{v_1 v_2}{c^2}} \qquad (25)$$

We obtain the Lorentz transformation between two generic frames. We see that Einstein velocity tend to c when v_2 tend to c independently of the value of v_1.

Conclusion

We consider the emission of light by a source moving with velocity v_1 through a frame where the speed of light is isotropic independent of the movement of the source, independent of v_1. Since the speed of light two-way in vacuum has the value c the one-way speed of light is also c in this frame. It is calculated the speed of light in the frame of the source and this speed is no more c. However the speed of light one-way that standard formulation affirm to be c is of course another "speed". This "speed" is Einstein speed of light that has been introduced by Einstein with other definition. The problem is the circularity of the definition associated to the notion of synchronization. We have shown that the one-way Einstein speed of light is c through the intrinsic desynchronization that relate synchronized clocks with Lorentzian clocks. The value of the one-way speed of light only is c in the *EF*. This frame is unique.

References

1. Mateljević M (2024) Lorentz Transformation and time dilatation. Ann Math Phys 7(1): 016-022. DOI: 10.17352/amp.000104
2. de Abreu, R. The physical meaning of synchronization and simultaneity in Special Relativity https://arxiv.org/abs/physics/0212020
3. de Abreu, R. Reinterpretation of Lorentz Transformation and Resolution of Relativity Paradoxes.

   http://arxiv.org/abs/physics/0203025; EPS-12 Trends in Physics, Abstracts p. 270, Budapest (2002). http://vixra.org/abs/1505.0065
4. de Abreu, R. Deduction of Lorentz Transformation from the existence of absolute rest. Deduction of the speed of light in any frame of reference, https://arxiv.org/abs/physics/0210023
5. Zbigniew Oziewicz, How do you add relative velocities? http://www.naturalphilosophy.org/pdf/group25.pdf
6. Zbigniew Oziewicz, Ternary relative velocity; astonishing conflict of the Lorentz group with relativity https://arxiv.org/abs/1104.0682
7. Kittel, C. Am. J. Phys. 42, 726 (1974) https://doi.org/10.1119/1.1987825
8. de Abreu, R. Gazeta de Física, vol. 21, Fasc. 3 (1998) https://www.spf.pt/magazines/GFIS/398/1153
9. Homem, G. Physics in a synchronized space-time. Master's thesis, Instituto Superior Técnico, Universidade Técnica de Lisboa, 2003

10. de Abreu, R. Ciência & Tecnologia dos Materiais, vol. 14, nº 1, p. 74 (2004)
https://vixra.org/abs/2301.0139

11. de Abreu, R and Guerra, V. [physics/0512196] Is the assumption of a special system of reference consistent with Special Relativity? (arxiv.org)

12. de Abreu, R. and Guerra, V. Relativity – Einstein's Lost Frame (Extra]muros[, Lisboa, 2005), 1st ed.
http://web.tecnico.ulisboa.pt/vguerra/papers/relshort.pdf

13. Guerra, V. and de Abreu, R. https://arxiv.org/abs/physics/0603258

14. Guerra, V. and de Abreu Foundations of Physics, On the Consistency between the Assumption of a Special System of Reference and Special Relativity Vol. 36, No. 12, December (2006) On the Consistency between the Assumption of a Special System of Reference and Special Relativity | SpringerLink

15. Guerra, V. and de Abreu, R. Eur. J. Phys. 26, 6 (2005)
The conceptualization of time and the constancy of the speed of light – IOPscience

16. de Abreu, R. and Guerra, V. Eur. J. Phys. 29, 1 (2007) The principle of relativity and the indeterminacy of special relativity - IOPscience https://iopscience.iop.org/article/10.1088/0143-0807/29/1/004.
The principle of relativity and the indeterminacy of special relativity (ulisboa.pt)

17. de Abreu, R. and Guerra, V. Eur. J. Phys. 30, 2 (2009)
Special relativity as a simple geometry problem - IOPscience
http://web.ist.utl.pt/d3264/papers/AG2009.pdf

18. de Abreu, R. and Guerra, V. EJTP Vol. 12 Issue 34, p183 (2015)
http://www.ejtp.com/articles/ejtpv12i34p183.pdf
Electronic Journal of Theoretical Physics (ejtp.com)

19. de Abreu, R. Guerra, V. http://vixra.org/abs/1804.0320

20. de Abreu, R. Guerra, V. http://vixra.org/abs/1805.0126

21. de Abreu, R. On the Experimental Determination of the One-Way Speed of Light
(PDF) Rodrigo de Abreu Departamento de Física, Instituto ...vixra.org/pdf/1808.0646v2.pdf · Departamento de Física, Instituto Superior Técnico Universidade Técnica de Lisboa, 1049-001 - PDFSLIDE.NET. https://vixra.org/abs/1808.0646

22. Einstein, A. Ann. Phys. 17, 132 (1905): "On the Electrodynamics of Moving Bodies", "Einstein's Miraculous Year, Five Papers That Changed the Face of Physics" Edited and Introduced by John Stachel, Princeton University Press (1998)

23. Consoli, M. Pluchino, A. Atti della Accademia Peloritana dei Pericolanti Vol. 96, No. S1, A2 (2018)

24. Kittel, C. Am. J. Phys. 42, 726 (1974). https://doi.org/10.1119/1.1987825

25. Andersen, F. PhD Thesis, NMBU (2017) https://nmbu.brage.unit.no/nmbu-xmlui/handle/11250/2500054

26. Myrstad, J. A. Borderology: Cross-disciplinary Insights from the Border Zone, p. 93, Springer (2019) **DOI**: 10.1007/978-3-319-99392-8_8 . http://dx.doi.org/10.1007/978-3-319-99392-8_8

27. Consoli, M. Pluchino, Michelson-Morley Experiments: An Enigma For Physics And The History Of Science, World Scientific (2019) https://www.worldscientific.com/worldscibooks/10.1142/11209

28. Consoli, M. Pluchino, A. The CMB, Preferred Reference System, and Dragging of Light in the Earth Frame, Universe (2021) https://doi.org/10.3390/universe7080311

29. Haug, E The Return of Absolute Simultaneity? A New Way of Synchronizing Clocks Across Reference Frames. http://vixra.org/abs/1605.0057

30. Haug, E A Motion Paradox from Einstein's Relativity of Simultaneity http://vixra.org/abs/1711.0408

31. de Abreu, R. Comment on "A Motion Paradox from Einstein's Relativity of Simultaneity" http://vixra.org/abs/1810.0452

32. Geloni, G. Kocharyan, V. Saldin, E. https://arxiv.org/abs/1601.07738

33. Geloni, G. Kocharyan, V. Saldin, E. https://arxiv.org/abs/1610.04139

34. Geloni, G. Kocharyan, V. Saldin, E. https://arxiv.org/abs/1704.01843

35. de Abreu, R. Guerra http://vixra.org/abs/1906.0312 Comment on "Misconceptions Regarding Conventional Coupling of Fields and Particles in XFEL Codes"

36. Mansouri, R. and Sexl, R. Gen. Relat. Gravit. 8, 497 (1977)

37. Zbigniew Oziewicz, Ternary relative velocity; astonishing conflict of the Lorentz group with relativity, Vladimir Gladyshev, Editor, Physical

Interpretations of Relativity Theory, Moscow 2007, pages 292-303, ISBN 978-5-7038-3178-6 https://arxiv.org/abs/1104.0682

38. Gianfranco Spavieri *et al* 2018 *J. Phys. Commun.* **2** 085009
39. Spavieri, G. PAIJ vol.1, Issue 1 (2017)
40. Consoli, M. Pluchino, A. Eur. Phys. J. Plus 133:295 (2018)
41. Burde, G. I. Special Relativity with a Preferred Frame and the Relativity Principle Journal of Modern Physics vol. 9, N° 8 (2018) https://doi.org/10.4236/jmp.2018.98100
42. Ricou, M. Physics Essays, Vol. 30, 461-468 (2017)
43. de Abreu, R. Guerra, V. http://vixra.org/abs/1906.0310
44. Wutke, A. https://www.researchgate.net/publication/326672264_Absolute_Simultaneity_and_Rest_Consistent_with_Special_Relativity_Science_or_Philosophy
45. Chandru Iyer 2018 *J. Phys. Commun.* **2** 118001
46. Potvin, G. Ether Interpretation of General Relativity, RESEARCHERS. ONE (2018) https://www.researchers.one/article/2018-12-7
47. Grøn, Ø. Eur. J. Phys. 27, 885 (2006) https://iopscience.iop.org/article/10.1088/0143-0807/27/4/019
48. Iyer, C. https://arxiv.org/abs/0811.0785
49. Iyer, C. Eur. J. Phys. 29, 4 (2008)
50. Iyer, C. Prabhu, G. Am. J. Phys. 78 (2) (2010)
51. Iyer, C. Prabhu, G. https://arxiv.org/abs/0710.1594
52. Moller, C. Proc. of The Royal Soc. A vol. 270, Issue 1342 (1960) https://doi.org/10.1098/rspa.1962.0220
53. Schwartz, H. M. Am. J. Phys. 39, 1269 (1971) https://doi.org/10.1119/1.1976621. A New Method of Clock Synchronization without Light Signals
54. Sears, F. W. Am. J. Phys. 37, 668 (1969) https://doi.org/10.1119/1.1975747. Simultaneity without synchronized clocks
55. Ramakrishnan, A. Journal of Math. Anal. and App. 249, 243 (2000)
56. Bricmont, J. https://arxiv.org/pdf/1703.00294.pdf. History of Quantum Mechanics or the Comedy of Errors

57. Giovanelli, M. https://www.researchgate.net/publication/338680431_Lorentz_Contraction_vs_Einstein_Contraction_Reichenbach_and_the_Philosophical_Reception_of_Miller'sEther-Drift_Experiments

58. Mamone-Capria, M. Journal for Foundations and Applications of Physics, vol.5, Nº 2, 163 (2018) [1704.02587] On the Incompatibility of Special Relativity and Quantum Mechanics (arxiv.org)

59. Spavieri, G. Gillies, G. T. Haug, E. G., Sanchez, A. (2019): Light propagation and local speed in the linear Sagnac effect, Journal of Modern Optics, http://doi: 10.1080/09500340.2019.1695005

60. Unnikrishnan, C. The Theories of Relativity and Bergson's Philosophy of Duration and Simultaneity During and After Einstein's 1922 Visit to Paris. *Preprints* **2020**,2020010273. http://doi: 10.20944/preprints202001.0273.v1

61. Spavieri, G. Guerra, V. de Abreu, R. Gillies, G. T Eur. Phys. J. D. 47, 457 (2008)

62. Spavieri, G. Gillies, G. Haug, E. G., Sanchez, A. Applied Physics Research Vol. 11, No. 4 (2019)

63. Spavieri, G. and Haug, E. G. Why the Sagnac effect favors absolute over relative simultaneity Physics Essays, Vol. 32, 331-337 (2019)

64. Spavieri, G. Gillies, G. T. Haug, E. G. Journal of Modern Optics, vol. 68, Issue 4 (2021)

65. Salmon, W., 1977. "The Philosophical Significance of the One-Way Speed of Light," *Noûs*, 11: 253–292

66. https://plato.stanford.edu/entries/spacetime-convensimul/

67. Perez, I. https://arxiv.org/abs/1102.4837 On the experimental determination of the one-way speed of light

68. Guerra, V. de Abreu, R. Comment on: "From classical ether-drift experiments: the narrow window for a preferred frame" [Phys. Lett. A 333 (2004) 355] Phys. Lett. A 361, Issue 6 (2007)

69. Lewis, G. F. Barnes, L. A. The One-Way Speed of Light and the Milne Universe Publication of the Astronomical Society of Australia, vol. 38, (2021) https://arxiv.org/abs/2012.12037

70. R.J. Buenker, GPS-Compatible Lorentz Transformation that Satisfies the Relativity Principle, Journal of Astrophysics and Aerospace Technology, 3: 115. DOI: 10.4172/2329-6542.1000115.

71. Netchitailo, V. S. https://vixra.org/abs/2012.0222

72. Pagano, A., Pagano, E.V. EPJ H 44, 321-330 (2019) https://doi.org/10.1140/epjh/e2019-90058-4

73. Greaves, E. D., Michel Rodrigues, A., Ruiz-Camacho, J. Am. J. Phys. 77, 894 (2009)

74. de Abreu, Guerra, V. Comment on "A one-way speed of light experiment" (2009) https://www.researchgate.net/publication/45886873

75. Leaf, B. Philosophy of Science, vol. 22, Number 1 (1955) https://doi.org/10.1086/287387

76. Spavieri, G. Quintero, J. Unnikrishnan, C. Gillies, G. Phys. Lett. A 376(s 6-7):795-797 (2012) DOI: 10.1016/j.physleta.2012.01.010

77. Scott Blair, G. Relativity and Indeterminacy. *Nature* **170,** 582 (1952). https://doi.org/10.1038/170582a0

78. Hines, C. Relativity and Indeterminacy. *Nature* **170,** 582 (1952). https://doi.org/10.1038/170582a0

79. Mayantz, L. The enigma of probability in physics https://inis.iaea.org/search/search.aspx?orig_q=RN:18074404

80. Mayantz, L Beyond the quantum paradox, Taylor & Francis (1994)

81. Medvedev, S. Yu. Progress in Physics, vol. 10, 151 (2014) http://www.ptep-online.com/2014/PP-38-04.PDF

82. Del Santo, F. Gisin, N. http://philsci-archive.pitt.edu/id/eprint/18601 (2021)

83. Fleming, A. Matveev, V. Matvejev, O. https://www.researchgate.net/publication/264851293_Self-field_Theory_and_General_Physical_Uncertainty_Relations

84. Hill, J. Cox, B. Proc. Royal Soc. A (2012) https://doi.org/10.1098/rspa.2012.0340

85. Leubner, C. Aufinger, K. Krumm, P. Eur. J. Phys. 13, 170 (1992)

86. Drągowski, M., Włodarczyk, M. The Doppler Effect and the Anisotropy of the Speed of Light. *Found Phys* **50,** 429–440 (2020). https://doi.org/10.1007/s10701-020-00337-5

87. Ronald Anderson, S. J., Stedman, G.E. Distance, and the conventionality of simultaneity in special relativity. *Found Phys Lett* **5,** 199–220 (1992). https://doi.org/10.1007/BF00692800

88. de Abreu, R. Simultaneity and Synchronization by Rods as a Simple Geometry Problem https://vixra.org/abs/2103.0196

89. Dias Ferreira, L. Criticism to the Twin´s Paradox, Colégio Valsassina, Lisbon, Portugal.
Universal Journal of Physics and Application Vol. 15(1), pp. 1 – 7,
DOI: 10.13189/ujpa.2021.150101 **Reprint (PDF)** (412Kb)

90. de Abreu, R. Comment on "Criticism to the Twin´s Paradox" by Luís Dias Ferreira, https://vixra.org/abs/2107.0077

91. Borah, B. K. Schwarzschild-like solution for the gravitational field of an isolated particle on the basis of 7- dimensional metric, International Journal of Scientific and Research, vol. 3, Issue 10, October 2013

92. Borah, B. K. An Approach to New Concept of Time on the Basis of Four Fundamental Forces of Nature, International Journal of Scientific and Research Publications, vol. 3, Issue 6, June 2013

93. Borah, B. K. Unification of Four Fundamental Forces of Nature Developing 7- Dimensional Metric on The Basis of New Concept of Time
https://www.worldwidejournals.com/international-journal-of-scientific-research- (IJSR)/article/unification-of-four-fundamental-forces-of-nature-developing-7andndash-dimensional-metric-on-the-basis-of-new-concept-of-time/NjU0QQ==/?is=1&b1=33&k=9

94. Homem, G. https://arxiv.org/abs/physics/0212050

95. de Abreu, R. The Energy-Entropy Principle
https://arxiv.org/abs/physics/0207022

96. de Abreu, R. Guerra, V. https://arxiv.org/abs/1203.2294

97. de Abreu, R. Guerra, V.
https://aapt.scitation.org/doi/10.1119/1.3698160

98. Caratheodory, C.
https://greatestgreeks.wordpress.com/2016/12/12/constantine-caratheodory/

99. Caratheodory, C.
https://www.researchgate.net/publication/312057731_On_Caratheodory's_approach_to_relativity_and_its_relation_to_hyperbolic_geometry

100. de Abreu, R. The Concept of Mass as a Quantity Derived from Motion.
https://vixra.org/abs/1505.0094

101. Mohazzabi, P. and Luo, Q. (2021) Has the Twin Paradox Really Been Resolved?. *Journal of Applied Mathematics and Physics*, **9**, 2187-2192.
doi: 10.4236/jamp.2021.99138.

102. de Abreu, R. Comment on "Has the Twin Paradox Really Been Resolved?", viXra.org e-Print archive, viXra:2209.0045

103. de Abreu, R. Simultaneity and Synchronization, the Preferred Frame and the Principle of Relativity, viXra.org e-Print archive, viXra:2204.0060

104. Reference:https://www.physicsforums.com/threads/potential-energy-formula- in-special-relativty.991687/

105. Reichenberger, Andrea (2018) The Clock Paradox: Luise Lange's Discussion REITCP-3 (philarchive.org)

106. Iyer, C. (2022) Importance of Synchronization in the observation of event coordinates DOI:10.36227/techrxiv.21657026

107. de Abreu, R. (2023) Proper Times, Time Dilation, Lorentz-FitzGerald Contraction and Distance, Time Ageing, Time Dilation-like and the Twin Paradox Conundrum Proper Times, Time Dilation, Lorentz-FitzGerald Contraction and Distance, Time Ageing, Time Dilation-like and the Twin Paradox Conundrum, viXra.org e-Print archive, viXra:2301.0029

108. de Abreu, R. (2023) The Time Dilation Conundrum and the Ageing of the Twins https://rxiv.org/abs/2301.0086

109. Choi, Yang-Ho (2022) Multiple velocity composition in the standard synchronization https://ui.adsabs.harvard.edu/abs/2022OPhy...20...17C/abstract

110. de Abreu, R. (2023) The Time Dilation Conundrum and the Ageing of the Twins https://rxiv.org/abs/2301.0086

111. de Abreu, R. (2023) The Resolution of the Twin Paradox with Three Frames of Reference - A Mathematical and Physical Report https://vixra.org/abs/2307.0052

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DOI 10.29083/HJ.48.04.2025/SC391

FOUNDATIONS: RELATIVISTIC QUANTUM THEORY I MATHEMATICAL THEORY

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Abstract

This is the first of a series on the foundations of quantum electrodynamics (QED). We first prove that: Minkowski's postulate is incompatible with the first postulate for two particles; analytic analysis of Dirac's equation leads to a new particle antiparticle separation, which reveals its time nonlocal nature; this also shows that the Pauli approximation is invalid for s-states and cut-offs are required before QED. We show Dirac's equation is not equivalent to the square root equation but is related by a unitary map. We then design the correct Hilbert space $KS^2[\mathbb{R}^n]$, for Feynman's path integral formulation of quantum theory and show it is not equivalent to $L^2[\mathbb{R}^n]$, the one for the Schödinger formulation, but is related by a unitary map. We also prove the existence of a universal clock and Santilli's dual of the real numbers to justify Feynman's view of antiparticles as particles going backwards in proper time. A major outcome of this study is that mathematical equivalence is not the same as physical equivalence and, careful physical analysis should be made before acceptance of mathematical concepts into physical theory.

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Introduction and Problems

We are ending the first quarter of the 21^{st} century and find ourselves still facing difficult problems left over from the 19^{th} century. Of particular concern, we have yet to obtain a relativistic version of Newtonian mechanics that is compatible with Maxwell's theory, and we can only partially claim a one-particle relativistic quantum theory (Dirac's equation requires holes for completion). The replacement of particles by fields have provided excellent agreement with experimental results. However, the methods used have led to an additional set of problems, that have defied resolution for almost five generations. The purpose of this study is to provide the mathematical foundations for a final solution. A major conclusion is that purely mathematical assumptions must receive careful vetting before acceptance as physically relevant and correct. The second paper is devoted to the classical foundations, while the third is devoted to the quantum foundations.

0.1 Framework and Historical Background

There are five postulates that provide the foundations for all of physical science:

(1) A real physical world, exists and is independent of our existence.

(2) This world is as it appears to us in our consciousness.

(3) This world is known via our senses (including surrogate instruments).

All the physical sciences are rooted in these postulates. Our scientific forefathers no longer exist, and yet, the laws they left us still exist (in some form), so the first postulate is clear. Without the second we can never be sure that our senses provide truth, making the third meaningless. Thus, we consider these postulates obvious.

0.1.1 Experimental Physics

Distortions and illusions about the world can appear in the consciousness of one or more us and not appear to others (the collective). Thus, we need a collective physical filter that allows us to distinguish physical truth from individual or group illusions. We refer to this filter as a scientific experiment.

Definition 0.1. *A scientific experiment is a controlled and reproducible experience. The basic postulate of experimental physics is that:*

(4) Objective reproducible empirical information about the physical world can be obtained through experimental investigation.

Definition 0.2. *We say that a physical subject is known when all possible cause effect-relations (of physical interest) can be a priori predicted under well-defined control conditions.*

0.1.2 Theoretical Physics

The objective of theoretical physics is to design and construct mathematical representations of the physical world. These representations must describe the cause-effect relationships observed in experiment, and must be physically and mathematically consistent, while using a minimal number of variables and parameters. The basic postulate of theoretical physics is as follows.

(5) Mathematics is the perfect intellectual design tool for the analysis and for certification of the consistency of representations of physical reality.

0.1.3 Newton-Maxwell Problem

The Newton-Maxwell problem was faced directly by Abraham, Einstein, Lorentz, Poincaré, Ritz, and other major thinkers of the 1900's. Starting with a complete analysis of the microscopic behavior of electrons and ions, Lorentz was able to explain all macroscopic properties of optics and electrodynamics (see [26], [27]). He was the first to obtain the transformations between different observers, showing how their relative velocities affected their results in different inertial frames. In his investigations, Poincaré discovered an error in the derivation and, after correction, named them Lorentz transformations. In addition, Poincaré observed that if time is treated as an imaginary coordinate, these Lorentz transformations form a group. (He also derived the proper time, which was later claimed by Minkowski.)

Einstein's work on the photo-electric effect, provided a different perspective. As noted by Brown [1], he did not believe that Maxwell's theory would survive the existence of photons, but considered Lorentz transformations fundamental. Thus, he derived them from kinematical arguments.

Einstein [6, 7] observed that a constant c appears in Maxwell's equations for all inertial observers. At that time, experimental information about the speed of light was limited, being restricted to macroscopic and astronomical studies. Einstein was the first to realize that a formal postulate on the

velocity of light is necessary. He proposed that all physical theories should satisfy the (now well-known) postulates of special relativity.

(**1**) The physical laws of nature and the results of all experiments were independent of the inertial frame of the observer (in which the experiment was performed).

(**2**) The speed of light in an empty space is constant and independent of the source or receiver motion (in an inertial frame).

Like others, Einstein realized that the idea of absolute space was neither knowable nor necessary. The first postulate abandons this idea. Einstein's second postulate appears to have abandoned the idea of absolute time.

Unlike Minkowski, Poincaré's insight and understanding of the difference between physics and mathematics helped him to resist the temptation to use the "physically unjustified" mathematical observation that time can be made a forth geometric coordinate as a (necessary) tool for the representation of physical reality. Minkowski later made this leap with much philosophical fanfare, but lacking any physical justification. Minkowski was a number theorist with few accomplishments in physics. In Walter's work, we find a clear discussion of Minkowski's physics background and his knowledge of Poincaré's work (see [46]).

Thus, we make explicit Minkowski's unacknowledged third postulate for Einstein's special theory of relativity.

(**3**) The correct implementation of the first two postulates requires that time be treated as a fourth coordinate, and the relationship between the components constrained to satisfy the natural invariance induced by the Lorentz group of electrodynamics (Minkowski space).

The physics community voted in favor of the Lorentz group as the proper transformation theory. Today, many believe that Newtonian theory is covered by Minkowski's version of the special theory. As will be shown later, the failure to conduct a complete (physical) analysis of the third postulate is at the root of all the major problems in the classical foundations.

0.2 Summary

In the first section we show that there are three representations of the proper time that follow from Einstein's version of the special theory. The first representation is well-known and is used to prove that Minkowski's postulate

is incompatible with Einstein's first postulate for two or more particles. We then discuss the center-of mass-problem due to Pryce [37], the many particle quantum problem discussed by Bakamjian and Thomas [2] and the no-interaction theorem first proven by Currie, Jordan, and Sudarshan [4]. We show that each of these foundational problems are a direct consequence of Minkowski's postulate. Finally, we prove that the second postulate is compatible with particle velocities larger than c, and that the assumption that the postulate holds for distant measurements is not valid.

In the second section, we briefly show that without the Minkowski postulate a new Klein-Gordon equation defined for all spins becomes possible. We then introduce an analytical separation method which shows that the Dirac equation can be separated directly revealing its time nonlocal properties. This approach allows us to show that the Pauli approximation cannot be used to study s-states in Hydrogen, and that cut-offs are required before QED. We conclude the section with a proof that the Dirac and square root equations although (mathematically) unitary equivalent, are not physically equivalent.

In the third section, we introduce the Henstock-Kurzweil integral (HK). It is very close to the one learned in calculus but extends the Lebesgue integral to include non-absolutely integrable functions (such as $e^{ix^2/2}$). We then design the natural Hilbert space for HK integrals, which allows us to construct the Feynman path integral in the manner he intended, while retaining both the physical intuition and computational advantage of his approach. This space $KS^2[\mathbb{R}^n]$ contains $L^2[\mathbb{R}^n]$ and the Schwartz distributions or generalized functions as continuous dense embeddings. From a physical point of view, the use of the HK-integral and $KS^2[\mathbb{R}^n]$ eliminates the extra time and effort required to first learn Lebesgue measure theory and then the theory of distributions.

In section four we use the third representation of proper time to prove the existence of a universal clock, under conditions weaker thaan the cosmological principle, which we call historical time (see [11] and [21]). We use this clock to construct Feynman's time-ordered operator calculus and use his theory to prove the last two remaining conjectures of Dyson concerning the foundations of QED.

1 Proper time and Minkowski's Postulate

Let m be the mass of a particle or the effective mass of a system of particles. We assume that the particle or system is defined in the phase space with variables $(t, \mathbf{x}, \mathbf{p})$ as seen by observer O, and variables $(t', \mathbf{x}', \mathbf{p}')$ as seen by observer O' (both inertial frames), where $\mathbf{x}, \mathbf{x}'$ is the position of the particle (or center of mass) and $(\mathbf{p}, \mathbf{p}')$ is the particle momentum (or center of mass momentum). If $\gamma^{-1}(\mathbf{v}) = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, where $\mathbf{v}$ $(\mathbf{v}')$ is the velocity of the particle, the first representation of the proper time is defined by our observers as.

$$d\tau = \gamma^{-1}(\mathbf{v})\, dt, \quad d\tau = \gamma^{-1}(\mathbf{v}')\, dt' \quad \underline{\text{proper time 1.}} \tag{1}$$

For the second representation, first square (1) to get:

$$d\tau^2 = dt^2 - \tfrac{1}{c^2} d\mathbf{x}^2, \quad d\tau^2 = dt'^2 - \tfrac{1}{c^2} d\mathbf{x}'^2.$$

Reaarrange terms to obtain:

$$d\tau^2 + \tfrac{1}{c^2} d\mathbf{x}^2 = dt^2, \quad d\tau^2 + \tfrac{1}{c^2} d\mathbf{x}'^2 = dt'^2 \Rightarrow$$
$$\sqrt{1 + \tfrac{\mathbf{u}^2}{c^2}}\, d\tau = dt, \ \mathbf{u} = \frac{d\mathbf{x}}{d\tau}, \quad \sqrt{1 + \tfrac{\mathbf{u}'^2}{c^2}}\, d\tau = dt', \ \mathbf{u}' = \frac{d\mathbf{x}'}{d\tau}.$$

If we let $b = \sqrt{c^2 + \mathbf{u}^2}$, $(b' = \sqrt{c^2 + \mathbf{u}'^2})$, then we have our second representation

$$cdt = bd\tau \quad \text{and} \quad cdt' = b'd\tau \quad \underline{\text{proper time 2.}} \tag{2}$$

For the third representation, use the relations $H = \gamma(\mathbf{v})\, mc^2$ for observer O and $H' = \gamma(\mathbf{v}')\, mc^2$ for observer O', to obtain:

$$d\tau = \tfrac{mc^2}{H} dt, \quad d\tau = \tfrac{mc^2}{H'} dt' \quad \underline{\text{proper time 3.}} \tag{3}$$

1.1 The Minkowski Problem

Minkowski used the first representation of proper time as a metric for his four-spacetime theory. As reported by Sommerfeld, Minkowski knew that the differential of proper time is not an exact one-form, as the value of τ depends on the forces acting on the particle (see the notes in [32]). Thus,

he introduced the co-moving observer as a substitute in order to use it as a metric. Einstein was the only one to openly object to Minkowski's postulate. He and Laub later published two papers on electrodynamics, which offered a different approach that was physically simpler, while not depending on the spacetime formalism (see [9], [10]). They argued that the spacetime formalism was complicated, required additional assumptions and did not add any new physics.

It is clear that Minkowski's postulate is mathematically very nice but it should have alarmed all physics thinkers, as "a co-moving observer cannot actually sit on the tangent plane of a bullet".

However, the idea of a four dimensional space-time was very attractive to mathematicians who had entered and dominated the field for over a decade. This mathematical authority had the effect of embedding Minkowski's postulate as natural part of Einstein's special theory. Consequently, intense physical investigation and justification normally given for new postulates did not occur. By the time problems in attempts to develop many particle relativistic quantum theories appeared, Minkowski's postulate was already sacred. For the general theory, Einstein only considered an approach that extended Minkowski's postulate (see Pais [31]). Unimpeachable physical problems only begin to appear after World War II, when new efforts to develop a many-particle theory were initiated.

1.1.1 The n-particle position Problem

In 1948 Pryce conducted the first study on the relativistic center-of-mass for two or more particles (see [37]). In order to reveal the first problem, we construct the many-particle clock for our O frame observer, we assume that n interacting particles have Hamiltonians H_i and a total Hamiltonian $H = \sum_{i=1}^{n} H_i$. The effective mass energy Mc^2, and total momentum $\mathbf{P}$, are defined as follows:

$$Mc^2 = \sqrt{H^2 - c^2\mathbf{P}^2}, \quad \mathbf{P} = \sum_{i=1}^{n} \mathbf{p}_i.$$

We can now also represent the Hamiltonian as $H = \sqrt{c^2\mathbf{P}^2 + M^2c^4}$. Pryce found that there three possible candidates for the center of mass. However, only one was canonical and available to all observers. The canonical center

of mass position $\mathbf{X}$ is defined by:

$$\mathbf{X} = \frac{1}{H}\sum_{i=1}^{n} H_i \mathbf{x}_i + \frac{c^2(\mathbf{S}\times\mathbf{P})}{H(Mc^2+H)}, \tag{4}$$

where $\mathbf{S}$ is the global spin of the system of particles relative to O (also see Longhi, Lusanna, and Pons [29]). Pryce then identified the first major problem, noting that the canonical representation of the center-of-mass cannot be the three-vector part of a four-vector. However, $\mathbf{X}$ is the canonical conjugate of $\mathbf{P}$, that is precisely the requirement for many-particle relativistic quantum theory. (It is clear that this conflicts with the third postulate.)

Five years after Pryce, Bakamjian and Thomas constructed a many-particle relativistic quantum theory using the canonical center of mass X. They explicitly showed that their theory satisfied Einstein's two postulates (see [2]). They conjectured that any attempt to enforce the third postulate would only be compatible with free particles (BT-conjecture).

In 1965 Currie, Jordan and Sudarshan [4], gave a proof of the BT-conjecture for two particles, which they renamed "The No-Interaction Theorem." It has since been extended to an arbitrary number of particles by Leutwyler [28].

Theorem 1.1. (No-Interaction Theorem) Suppose we have a system of n particles with phase space variables $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^{n}$ defined on $\mathbb{R}^{3n}\times\mathbb{R}^{3n}$ with the following properties:

1. The system provides a Hamiltonian representation.
2. The system has a canonical representation of the Poincaré group.
3. Each $\mathbf{x}_i$ is the vector of the four-vector.

Then these assumptions are only compatible with free particles.

Thus, we see that Minkowski's postulate appears to be the culprit that has hindered progress in both classical and quantum mechanics for over 120 years. The first hint that this may be the problem was actually given by Feynman, who showed that by allowing time to recover its role in physics as it appears in our consciousness, he could construct his operator calculus, his formulation of quantum mechanics, and QED. Feynman was criticized, but showed that this approach allowed him to give a physically intuitive and

a computationally efficient way of approaching QED. Although he shared the Nobel prize with Schwinger and Tomongona, for their work on the same problems, his approach is now taught in programs all over the world. The following theorem shows the importance of an early investigation of Minkowski's postulate.

Theorem 1.2. (Minkowski Incompatible Theorem) *The addition of Minkowski's postulate to the postulates of Einstein is incompatible for two or more particles,*

Proof. Let O and O' be two inertial observers. Without loss, we can assume that both clocks begin when their origins coincide and O' is moving with uniform velocity $\mathbf{v}$ as seen by O. Let two particles, each the source of an electromagnetic field, move with velocities $\mathbf{w}_i\,(i=1,2)$, as seen by O, and $\mathbf{w}'_i\,(i=1,2)$, as seen by O', such that:

$$\begin{aligned}\mathbf{x}'_i &= \mathbf{x}_i - \gamma(\mathbf{v})\,\mathbf{v}t + [\gamma(\mathbf{v})-1]\left(\mathbf{x}_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v},\\ \mathbf{x}_i &= \mathbf{x}'_i + \gamma(\mathbf{v})\,\mathbf{v}t' + [\gamma(\mathbf{v})-1]\left(\mathbf{x}'_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v} \quad \text{and,}\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{w}'_i &= \mathbf{w}_i - \gamma(\mathbf{v})\,\mathbf{v} + [\gamma(\mathbf{v})-1]\left(\mathbf{w}_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v},\\ \mathbf{w}_i &= \mathbf{w}'_i + \gamma(\mathbf{v})\,\mathbf{v} + [\gamma(\mathbf{v})-1]\left(\mathbf{w}'_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v}.\end{aligned} \tag{6}$$

Thus, there is no problem in requiring the positions and velocities to be transformed as expected. However, when we try to transform the clocks, we see the problem at once because

$$t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_1\cdot\mathbf{v}/c^2\right) \quad \text{and} \quad t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_2\cdot\mathbf{v}/c^2\right). \tag{7}$$

This is impossible unless $\mathbf{x}_1\cdot\mathbf{v} = \mathbf{x}_2\cdot\mathbf{v}$ (all times). This means that the two observers cannot use their clocks to share information (with other observers) regarding two or more particles. It follows that if we attempt to replace $\mathbf{x}_i$ and $\mathbf{x}'_i$ with four vectors using t and t', then the first postulate fails. Thus, these three postulates are mathematically and physically incompatible. □

Remark 1.3. *The Lorentz transformations (equations (1.4) (1.5) and (1.6)) contain information about each observer and each particle, so it is not clear that they can be converted into the tensor notation that came into fashion after Sommerfeld simplified the (truly) complicated notation of Minkowski.*

1.2 Proper Time II and Einstein Compatibility Theorem

In this section, we first demonstrate how the second definition of proper time avoids the problems created by Minkowski's postulate.

1.2.1 One-Particle Clock

Assume a system of n-particles are observed by O, who is able to identify each particle and attach a vector $\mathbf{x}_i$ to the i^{th} particle, denoting its spacial distance to the origin. If $\mathbf{w}_i$ is the velocity of particle i as indicated by O, let $\gamma^{-1}(\mathbf{w}_i) = \sqrt{1 - \mathbf{w}_i^2/c^2}$. The i^{th} particle proper time is defined as

$$d\tau_i = \gamma^{-1}(\mathbf{w}_i)dt, \quad \mathbf{w}_i = \frac{d\mathbf{x}_i}{dt}, \quad d\tau_i^2 = dt^2 - \tfrac{1}{c^2}d\mathbf{x}_i^2. \tag{8}$$

Rewrite the last term to get:

$$dt^2 = d\tau_i^2 + \tfrac{1}{c^2}d\mathbf{x}_i^2, \Rightarrow cdt = \left(\sqrt{\mathbf{u}_i^2 + c^2}\right) d\tau_i, \quad \mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau_i} = \gamma(\mathbf{w}_i)\mathbf{w}_i. \tag{9}$$

If we let $b_i = \sqrt{\mathbf{u}_i^2 + c^2}$, the second term in equation (9) becomes $cdt = b_i d\tau_i$. This leads to the following new identity:

$$\frac{1}{c}\frac{d}{dt} \equiv \frac{1}{b_i}\frac{d}{d\tau_i}, i = 1, \ldots, n. \tag{10}$$

Thus, we obtained an interesting set of relationships between the proper time of each particle and that of the observer. Applying this to $\mathbf{x}_i$, we obtain another set of relationships showing that the tangent space is left invariant in all cases.

$$\frac{\mathbf{w}_i}{c} = \frac{1}{c}\frac{d\mathbf{x}_i}{dt} \equiv \frac{1}{b_i}\frac{d\mathbf{x}_i}{d\tau_i} = \frac{\mathbf{u}_i}{b_i}, i = 1, \ldots, n. \tag{11}$$

The new particle coordinates are $(\mathbf{x}_i, \tau_i)$. In this representation, $\mathbf{x}_i$ is uniquely defined relative to O, whereas τ_i is uniquely defined for each particle.

Remark 1.4. *Equation (11) shows that the second postulate is compatible with speeds higher then the speed of light as seem by inertial observers. It also means that the use of $\beta = \mathbf{w}_i/c$ as a parameter for distant measurements is no-longer reliable and that time dilation is a physically incorrect concept.*

As the momentum is $\mathbf{p}_i = m_i\gamma(\mathbf{w}_i)\mathbf{w}_i$ and $\mathbf{u}_i = \gamma(\mathbf{w}_i)\mathbf{w}_i$, we can also represent the momentum as $\mathbf{p}_i = m_i\mathbf{u}_i$. It follows that the particle phase space is also left invariant. Because m_i is the rest mass, we see that the concept of relativistic mass increase is also a physically incorrect concept.

Theorem 1.5. Einstein Compatibility Theorem *If the observer time is used at the global level and each particle's proper time is used at the local level, then Einstein's postulates hold and the transformations preserving the first postulate are:*

$$b_i{}' = \gamma(\mathbf{v})\left(b_i - \mathbf{u}_i \cdot \mathbf{v}/c\right)$$

and

$$b_i = \gamma(\mathbf{v})\left(b_i' + \mathbf{u}'_i \cdot \mathbf{v}/c\right).$$

Proof. If we use each particle proper time then, for observer O'

$$\frac{\mathbf{w}_i'}{c} = \frac{1}{c}\frac{d\mathbf{x}_i'}{dt} \equiv \frac{1}{b_i'}\frac{d\mathbf{x}_i'}{d\tau_i} = \frac{\mathbf{u}_i'}{b_i'}, i = 1, \ldots, n.$$

From $t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_i \cdot \mathbf{v}/c^2\right)$ we have $cdt' = \gamma(\mathbf{v})\left(cdt - d\mathbf{x}_i \cdot \mathbf{v}/c\right)$. Using the two identities $cdt' = b_i' d\tau_i$ and $cdt = b_i d\tau_i$, we obtain:

$$b_i{}' d\tau_i = \gamma(\mathbf{v})\left(b_i d\tau_i - d\mathbf{x}_i \cdot \mathbf{v}/c\right)$$

Dividing by $d\tau_i$, we see that the transformations between b_i and b_i' are

$$b_i{}' = \gamma(\mathbf{v})\left(b_i - \mathbf{u}_i \cdot \mathbf{v}/c\right).$$

By a similar calculation, we obtain the reverse relationship:

$$b_i = \gamma(\mathbf{v})\left(b_i' + \mathbf{u}'_i \cdot \mathbf{v}/c\right).$$

It now follows that the two postulates are valid for all observers. □

Each observer can now use their clock and the canonical center of mass to quantize the n-particle system (following [2]).

Remark 1.6. *We note that the second postulate is still valid for all observers, and is necessary for observers to share information, insuring the first postulate.*

2 Quantum Problems

This section is devoted to the analysis of a number of missed and/or misunderstood mathematical issues in the foundations of relativistic quantum theory. We begin with a new look at the Dirac equation and investigate what happens when the potential is viewed as an addition to the mass (as often occurs classically). This leads us to a new Klein-Gordon equation, without changing the wave function and is compatible with all particle spin values. We then conduct an analytical analysis of the Dirac equation and show that it can be diagonalized exactly without changing the wave function (in distinction to the Foldy-Wouthuysen method [13]). It allows us to directly demonstrate that the Pauli approximation cannot be used to study the s-states in hydrogen. We finally show that although related by a unitary transformation, the Dirac and square root operators are not physically equivalent. Thus, mathematical equivalence may have nothing to do with the physics.

2.1 The Dirac Equation

After Palui proposed his exclusion principle in 1925 ([35]), Dirac was able to find the first physically viable method around the square-root problem in 1926 (see [5]). Raised in the British tradition of quaternions, Dirac knew that quaternions could be used to generate a spin-1/2 representation for the rotation group. He used Pauli matrices to write the square-root operator as a perfect square:

$$\sqrt{c^2\mathbf{p}^2 + m^2c^4} = \sqrt{\left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2\right]^2}, \quad \text{where,} \quad \beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2. \end{bmatrix} \tag{12}$$

and the matrix $\boldsymbol{\alpha}$ is defined by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, with

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus obtaining an alternative representation of equation (12), now known as the Dirac equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2\right]\Psi \tag{13}$$

We must now view Ψ as a vector-valued function or spinor. In other words, $\Phi \in L^2(\mathbb{R}^3, C^4) = L^2(\mathbb{R}^3) \otimes C^4$ is a four-component column vector $\Psi = (\psi_1, \psi_2, \phi_1, \phi_2)^t$. In this representation, $\psi = (\psi_1, \psi_2)^t$ is the particle component and $\phi = (\phi_1, \phi_2)^t$ is the antiparticle component. Dirac's equation provided the correct spin value for the electron and the known (at that time) spectrum for hydrogen. His equation also predicted negative energy solutions, which led Dirac to propose the hole theory of vacuum. In this view, a hole is an electron with a negative energy and positive charge. This means that Dirac's theory cannot be applied to a single particle. However, the later discovery of the positron confirmed this approach. Schweber [45] provides a good discussion of the difficulties that eventually lead to QED.

2.1.1 The Dirac-Klein-Gordon Connection

When the mass is constant it was shown by Case [3] that a Foldy-Wouthuysen (unitary) transformation can be constructed for particles of spin 0 and 1, Then Pursey [34] showed that such a transformation existed for all values of spin. This means that we can obtain a Dirac equation for all spin values. In this section, we show that, unconstrained by the Minkowski postulate, there is a new and important relationship between the Dirac and Klein-Gordon equation. Let $E[\Psi] = [c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta mc^2 + V]\,\Psi$, where $V = e\phi$, $\partial V/\partial t = 0$, is the potential energy and $\boldsymbol{\pi} = \mathbf{p} - e/c\mathbf{A}$. Now note that $\beta mc^2 + V = \beta(mc^2 + \beta V)$, such that

$$\left[E - c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} - \beta(mc^2 + \beta V)\right] \Psi = \mathbf{0} \tag{14}$$

Now multiply on the left by $[E + c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta(mc^2 + \beta V)]$, doing the standard computations:

$$e\boldsymbol{\alpha} \cdot [E\mathbf{A} - \mathbf{A}E] = ie\hbar\boldsymbol{\alpha} \cdot \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{and}$$

$$c\beta\boldsymbol{\alpha} \cdot [\beta V\mathbf{p} - \mathbf{p}\beta V] = ie\hbar c\boldsymbol{\alpha} \cdot \nabla\phi,$$

to obtain:

$$\begin{aligned} &\left\{E^2 - c^2\boldsymbol{\pi}^2 + e\hbar c\boldsymbol{\Sigma} \cdot \mathbf{B} + ie\hbar c\boldsymbol{\alpha} \cdot \mathbf{E}\right. \\ &\left. +2i\hbar c\boldsymbol{\alpha} \cdot \nabla V - 2cV\boldsymbol{\alpha} \cdot \mathbf{p} - (mc^2 + \beta V)^2\right\} \Psi = \mathbf{0}. \end{aligned} \tag{15}$$

Where $\mathbf{E} = -\partial\mathbf{A}/\partial ct - \nabla\phi$ is the electric field. We thus obtain a Klein-Gordon equation valid for all values of spin:

$$\begin{aligned} -\hbar^2\frac{\partial^2\Psi}{\partial t^2} = \{c^2\boldsymbol{\pi}^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} - ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} \\ -2i\hbar c\boldsymbol{\alpha}\cdot\nabla V - 2cV\boldsymbol{\alpha}\cdot\mathbf{p} + (mc^2+\beta V)^2\}\,\Psi. \end{aligned} \tag{16}$$

When $\mathbf{A}$ and V are zero, the above becomes the standard KG-equation. Belief in the correctness of the Minkowski postulate dictated the current manner of introducing minimal coupling. This approach is discussed in Schiff ([40], see pg. 320 eq. 42.9 and pg. 329 eq. 43.25). Schweber provides a good discussion of the interpretation problems because of the possible negative values of the natural probability density (see [44]).

2.1.2 Analytical Analysis of the Dirac Equation

A complete understanding of the Dirac equation was only possible after the development of the geometric algebra. We now see that Dirac used Lorentz-invariant Clifford algebra to represent the complex algebra of observables for the electron (see Hestenes [20]). However, it is apparently believed that when interaction is present, a complete separation of the particle and antiparticle components of his equation without approximation is not possible. The algebraic studies of Foldy-Wouthuysen [13], Feynman and Gell-Mann [12], and Pauli [36] along with the various approximation approaches in the literature, may have supported this idea.

The first exact analytical separation of Dirac's equation, with minimal coupling, was reported by Gill et. al. (2005). Because this is important for our physical interpretation of Dirac's equation, the correct physical interpretation of the zitterbewegung, the correct physical relation between Dirac's equation and the square root equation and to understand the physical failure of the Palui approximation for the study of s-states in hydrogen, we provide a brief outline of this method (see [19] for details).

If $\mathbf{A}(\mathbf{x},t)$ and $V(\mathbf{x})$ are vector and scalar potentials, respectively with standard identification $\mathbf{p} \to \boldsymbol{\pi} = \mathbf{p} - (e/c)\mathbf{A}$ and write (13) in a two-component form:

$$i\hbar\frac{\partial\psi}{\partial t} - \left(mc^2+V\right)\psi = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\varphi, \tag{17}$$

and

$$i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}=c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi} \tag{18}$$

where the vectors $\boldsymbol{\psi}=[\psi_1,\psi_2]^t$ and $\boldsymbol{\varphi}=[\varphi_1,\varphi_2]^t$ represent the particle and antiparticle components $\Psi=[\boldsymbol{\psi},\boldsymbol{\varphi}]^t$. These two equations can be separated using the method of integrating factors (as in elementary differential equations). We solve for $\boldsymbol{\varphi}$ in the second equation and use the result in the first to obtain an equation that depends only on $\boldsymbol{\psi}$. Thus, we need to determine δ such that.

$$e^{-\delta t}i\hbar\frac{\partial}{\partial t}\left[\boldsymbol{\varphi}e^{\delta t}\right]=i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}.$$

The left-hand side was simplified and compared to obtain that $\delta=\frac{1}{i\hbar}\left(mc^2-V\right)$. We now assume that $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ vanish at $-\infty$ and integrate to obtain.

$$\begin{aligned}&i\hbar\frac{\partial\boldsymbol{\psi}}{\partial t}-\left(mc^2+V\right)\boldsymbol{\psi}=\\&c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left\{\int_{-\infty}^{t}\exp\left\{\tfrac{i}{\hbar}\left(mc^2-V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi}(\mathbf{x},\tau)d\tau\right\}.\end{aligned} \tag{19}$$

Using the same method, we obtain our second equation for $\boldsymbol{\varphi}$:

$$\begin{aligned}&i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}=\\&c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left\{\int_{-\infty}^{t}\exp\left\{-\tfrac{i}{\hbar}\left(mc^2+V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\pi\right)\boldsymbol{\varphi}(\mathbf{x},\tau)d\tau\right\}.\end{aligned} \tag{20}$$

Equations (19) and (20) now decomposed $L^2(\mathbb{R}^3,C^4)$ into a direct sum $L^2(\mathbb{R}^3,C^4)=L^2(\mathbb{R}^3,C^2)\oplus L^2(\mathbb{R}^3,C^2)$. For later use, the probability density for $\boldsymbol{\psi}$ is:

$$\rho_{\boldsymbol{\psi}}(t)=\left|\boldsymbol{\psi}\left(\mathbf{x},t\right)\right|^2+\left|c\int_{-\infty}^{t}\exp\left\{\tfrac{i}{\hbar}\left(mc^2-V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi}\left(\mathbf{x},\tau\right)d\tau\right|^2. \tag{21}$$

Thus, we obtain a complete analytic diagonalization of the particle and antiparticle wave functions (see [19]). The decomposition is analytic compared to Foldy and Wouthuysen's group theoretical framework in that the wave function is not transformed (see [13]). We discuss this in further detail, after studying the square root operator.

2.1.3 Pauli Approximation

For the eigenvalue problem, equations (17) and (18) can be written as.

$$\left(E - V - mc^2\right)\psi = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\varphi,$$
$$\left(E - V + mc^2\right)\varphi = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\psi.$$

Using

$$\varphi = c\left(E - V + mc^2\right)^{-1}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\psi,$$

in the first equation and

$$\psi = c\left(E - V - mc^2\right)^{-1}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\varphi,$$

in the second equation. After simplification, we obtain:

$$\left(E - V - mc^2\right)\psi = \frac{c^2\left(\boldsymbol{\sigma}\cdot\mathbf{p}V\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V + mc^2\right)^2}\psi + \frac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V + mc^2\right)}\psi \qquad (22)$$

and

$$\left(E - V + mc^2\right)\varphi = \frac{c^2\left(\boldsymbol{\sigma}\cdot\mathbf{p}V\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V - mc^2\right)^2}\varphi + \frac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V - mc^2\right)}\varphi. \qquad (23)$$

Equations (22) and (23) first appeared in Williams in 1940 (see [47]). He was a student of Slater, who included them in the appendix of his 1960 book (see [41]). We call these Williams-Slater equations. As will be seen, they are important for understanding the mathematical foundations of QED. In equation (22), $\left(E - V + mc^2\right)^{-2}$ is replaced by 0 and $E - V + mc^2$ is approximated by $2mc^2$ to obtain the Pauli's equation:

$$\left(E - V - mc^2\right)\psi = \frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2m}\psi.$$

The probability density for this equation now becomes:

$$\rho_\psi = \left|\psi\left(\mathbf{x}\right)\right|^2 + \left|\frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2mc}\psi\left(\mathbf{x}\right)\right|^2$$

If we use

$$\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right) = \pi^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\mathbf{B},$$

We obtain the standard form:

$$E\psi = \left[\frac{\boldsymbol{\pi}^2}{2m} - \frac{e\hbar(\boldsymbol{\sigma}\cdot\mathbf{B})}{2mc} + V + mc^2\right]\psi \tag{24}$$

In the case of interest, $V_0 = -mc^2 r_0/r$, $\mathbf{A} = \boldsymbol{\mu}_p \times \mathbf{r}/r^3$, with the spin oriented along the z-axis; thus, $A_r = A_\theta = 0$ and $A_\phi = \frac{2\mu_p s_p \sin\theta}{r^2}$. Therefore, we see that $\boldsymbol{\pi}^2 = \mathbf{p}^2 + \mathbf{A}^2$. Recall that QED provides corrections to the Dirac equation for s-states in hydrogen based on the Pauli approximation. Thus, it is important to know the extent to which we can draw any conclusions from the Pauli equation regarding the behavior of the Dirac equation in s-states. Recalling that a Dirac electron has a finite probability of being at the origin in the s-state, we see that the following invalidates Pauli's approximation.

1. $(E - V + mc^2)$ cannot be approximated by $2mc^2$ for s-states.
2. Because $\mathbf{A}$ has an r^{-2} term and $\mathbf{A}^2$ has an r^{-4} term they cannot be omitted for s-state calculations.
3. The use of $\rho_\psi = |\psi(\mathbf{x})|^2$ because this as the density is incorrect and the corrected density $\rho_\psi = |\psi(\mathbf{x})|^2 + \left|\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})}{2mc}\psi(\mathbf{x})\right|^2$ cannot be used because of (2).

To resolve these difficulties, we note that the binding energy in hydrogen is 13eV, while the rest mass of the electron is 5×10^5eV, so the ratio is 2.6×10^{-5}. Thus, we can replace $E - V + mc^2$ with $2mc^2 + \frac{e^2}{r}$ with little error, but a cut-off is required (see [19] for details). The important conclusion is that any analysis of the Dirac hydrogen atom problem for s-states requires a cut-off prior to any QED calculations.

Remark 2.1. *We note that, for the p-state each of these terms are small, so the Pauli approximation is valid for p-state calculations*

2.1.4 The Square Root Equation

In 2005, Gill and Zachary [18] used the theory of fractional powers of linear operators to construct an (analytic) representation of the square-root energy operator valid for all spin values. A better representation was obtained by

Samko (see [38] pg. 270). If Δ is the Laplacian and $\omega = mc/\hbar$, we have:

$$\begin{aligned} &\beta\hbar c\left[\sqrt{\omega^2 I - \Delta}\right]\psi(\mathbf{x},t) = \beta mc^2\psi(\mathbf{x},t) \\ &-\frac{\beta\hbar c}{2\pi^2}\left\{\int_{\mathbb{R}^3}\frac{K_2[\omega|\mathbf{x}-\mathbf{y}|]}{|\mathbf{x}-\mathbf{y}|^2}\left[\psi(\mathbf{x},t)-\psi(\mathbf{y},t)\right]d\mathbf{y}\right\}. \end{aligned} \tag{25}$$

The integral is over $\mathbb{R}^3$ showing in what sense the square root operator is nonlocal in space. For a better understanding of this extended object, we require more information about the Bessel functions $K_2[\omega|\mathbf{y}|]$, $K_1[\omega|\mathbf{y}|]$, $K_{1/2}[\omega|\mathbf{y}|]$ and $K_0[\omega|\mathbf{y}|]$ (see Gradshteyn and Ryzhik [15]).

$$\begin{aligned} &K_2[\omega|\mathbf{y}|] = K_0[\omega|\mathbf{y}|] + \frac{K_1[\omega|\mathbf{y}|]}{[\omega|\mathbf{y}|]} \quad \text{and} \\ &\frac{K_{1/2}[\omega|\mathbf{y}|]}{[\omega|\mathbf{y}|]^{1/2}} = \sqrt{\frac{\pi}{2}}\frac{exp\{-\omega|\mathbf{y}|\}}{[\omega|\mathbf{y}|]}. \end{aligned}$$

Since the integral of $|\mathbf{x}-\mathbf{y}|^{-2}$ is finite, the effective singularity properties for $0 < z \ll 1$ are:

$$\left.\begin{aligned} \tfrac{K_1[z]}{z} &= C_1[1+\theta_1(z)]z^{-2} \\ \tfrac{K_{1/2}[z]}{z^{1/2}} &= \sqrt{\tfrac{\pi}{2}}[1+\theta_2(z)]z^{-1} \\ K_0[z] &= C_0[1+\theta_0(z)]\ln z^{-1} \end{aligned}\right\}.$$

We see that K_1 diverges at 0 like z^{-2}. We conjecture that this term is the particle representation for the strong interaction. K_0 is an an integrable singularity. (We include $K_{1/2}$ for comparison, as it is essentially the well-known Yakawa potential, expected to account for short-range nuclear interactions). The effective properties for $z \gg 1$ are:

$$\left.\begin{aligned} \tfrac{K_1[z]}{z} &= C_1[1+\Theta_1(z)]z^{-3/2}e^{-z} \\ \tfrac{K_{1/2}[z]}{z^{1/2}} &= \sqrt{\tfrac{\pi}{2}}\left[1+\Theta_{1/2}(z)\right]z^{-1}e^{-z} \\ K_0[z] &= C_0[1+\Theta_0(z)]z^{-1/2}e^{-z} \end{aligned}\right\}.$$

In this region, we have a cut-off range and the strength of the interaction reverses, with the K_0 having the longest tail. We conjecture that this term is the particle representation for the weak interaction. It is clear that the square root operator is nonlocal in space and can be treated as an identity operator outside a few Compton wavelengths.

2.1.5 Discussion I

The important conclusion from the previous two sections is that the Dirac and square root operators are not physically equivalent. The Dirac operator is nonlocal in time, whereas the square root operator is nonlocal in space. It is shown in [19] that the correct physical interpretation of the zitterbewegung and the fact that the expected instantaneous value of the velocity measurement of a Dirac particle is $\pm c$ is that it instantaneously jumps between $+c$ and $-c$ at each point in time to make it appear as a point in space.

We must now account for the fact that the Foldy-Wouthuysen is a unitary transformation mapping the coupled particle-antiparticle Dirac operator to the uncoupled particle-antiparticle square root operator. From equations (2.9) and (2.10), we see that the Foldy-Wouthuysen unitary operator maps a nonlocal-in-time operator to a nonlocal-in-space operator, so that the unitary property of an operator is not sufficient to give confidence in a physical interpretation. We should also remember that the concept of unitary equivalence is purely mathematical and taken as a tool for physics without complete physical justification.

Remark 2.2. *The Schödinger representation (on $L^2[\mathbb{R}^3]$) and the Heisenberg representation $\ell_2[\mathbb{R}^3]$ were known to be physically equivalent before they were proven to also be mathematical equivalent (i.e, related by a unitary mapping) by von Neumann. However, there are an infinite number of Hilbert spaces related to $\ell_2[\mathbb{R}^3]$ by a unitary mapping, which have nothing to due with quantum theory. In a physically justified sense, $L^2[\mathbb{R}^3]$ is not the correct Hilbert space for Feynman's path integral formulation of quantum mechanics. This is discussed in the next section. This space serves as another counter example to the notion that mathematical equivalence means physical equivalence.*

3 Hilbert Space for Feynman's Formulation of Quantum Mechanics

We begin this section with a brief introduction to the Henstock-Kurzweil integral (HK). This integral is well defined for operator-valued functions that may not be separably valued (where both the Bochner and Pettis integrals are undefined). Intuitively, one uses a version of the Riemann integral (of calculus) with the interior points chosen first, whereas the size of the base

rectangle around any interior point is determined by an arbitrary positive function defined at that point. This integral was discovered independently by Henstock [22] and Kurzweil [25]. Our aim is to provide a sense of the conceptual and technical simplicity of the HK integral. Those interested in more details and proofs are directed to Gill and Zachary [17]. Let $\mathcal{H}$ be a separable Hilbert space and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let $[a,b] \subset \mathbb{R}$ and, for each $t \in [a,b]$, let $H(t) \in L(\mathcal{H})$ be a given family of operators.

Definition 3.1. *Let $\delta(t)$ map $[a,b] \to (0,\infty)$, and let $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$, where $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$. We call $\mathbf{P}$ a HK-δ partition for $[a,b]$ provided that for $0 \leqslant i \leqslant n-1$, $t_i, t_{i+1} \in (\tau_{i+1} - \delta(\tau_{i+1}), \tau_{i+1} + \delta(\tau_{i+1}))$.*

Definition 3.2. *The family $H(t)$, $t \in [a,b]$, is said to have a (uniform) HK integral if there is an operator $Q[a,b]$ in $L(\mathcal{H})$ such that, for each $\varepsilon > 0$, there exists a function δ from $[a,b] \to (0,\infty)$ such that, whenever $\mathbf{P}$ is a HK-δ partition for $[a,b]$, then*

$$\left\| \sum_{i=1}^{n} \Delta t_i H(\tau_i) - Q[a,b] \right\| < \varepsilon.$$

In this case, we write

$$Q[a,b] = (HK) \int_a^b H(t)dt.$$

Example 3.3. *The following example shows that there are HK-integrable functions, that are not Lebesgue integrable. If $F'(t) = 2t\sin(1/t^2) - (2/t)\cos(1/t^2)$, then for $t \in (0,1)$ and $F'(0) = 0$. It is easy to see that $F(t) = t^2\sin(1/t^2)$, so that*

$$(HK) \int_0^1 \left(2t \sin \frac{1}{t^2} - 2\frac{1}{t} \cos \frac{1}{t^2} \right) dt = \sin 1.$$

The following theorem explains the difference between the two integrals.

Theorem 3.4. *Let $f(t) : [a,b] \to \mathbb{R}$.*

1. *If $f(t)$ is L-integrable on $[a,b]$, then it is HK-integrable on $[a,b]$ and: HK-$\int_a^b f(t)dt$ = L-$\int_a^b f(t)dt$.*

2. *If $f(t)$ is HK-integrable on $[a,b]$, then $\sup_{t>a} \left| \int_a^t f(s)ds \right| < \infty$.*

3.1 The KS-Hilbert Space

In this section, we construct a particular (separable) Hilbert space $KS^2[\mathbb{R}^n]$. This space is designed to contain the class of HK-integrable functions, the space of the test functions $\mathcal{D}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, $\mathfrak{M}[\mathbb{R}^n]$ and each of the $L^p[\mathbb{R}^n]$ spaces, for $1 \leqslant p \leqslant \infty$. Furthermore, each of the latter class spaces will be contained in $KS^2[\mathbb{R}^n]$ as a continuous dense and compact embeddings (e.g., weakly convergent sequences in each of the above spaces are strongly convergent in $KS^2[\mathbb{R}^n]$). In particular, $KS^2[\mathbb{R}^n]$ is perfect for constructing Feynman's path integral formulation of quantum mechanics (in the form close to his suggestion, while maintaining the physical intuition and computational advantages).

3.2 The Inner Product

It is not well-known in the mathematical community, but there are two inner products that may be defined on the class of functions $\mathbb{C}_c^\infty[\mathbb{R}^n]$, $n \in \mathbb{N}$. The first is well-known, if μ is a measure on $\mathbb{R}^n$, we have:

$$(f,g)_1 = \int_{\mathbb{R}^n} f(\mathbf{x})g^*(\mathbf{x})d\mu(\mathbf{x}),$$

and

$$(f,g)_2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\mathbf{x})g^*(\mathbf{y})d\mu(\mathbf{x})d\mu(\mathbf{y}).$$

In the first case,

$$||f||_1^2 = \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d\mu(\mathbf{x}), \tag{26}$$

while in the second case,

$$||f||_2^2 = \left| \int_{\mathbb{R}^n} f(\mathbf{x})d\mu(\mathbf{x}) \right|^2, \tag{27}$$

It is clear that (26) is a norm for absolutely integrable functions, while (27) is a norm for non-absolutely integrable functions. This latter form of the inner product was first (indirectly) revealed in the work of Kuelbs [24].)

3.3 The Kuelbs-Steadman Hilbert space $KS^2[\mathbb{R}^n]$

We know from the work of Henstock [22] that, in the one-dimensional case we can norm the non-absolutely integrable functions on $[0, \infty)$ by:

$$||f|| = sup_{t>0} \left| \int_0^t f(s)ds \right|$$

and the closure in this norm contains some distributions but no other information is known. Thus, we imposed the following (physically motivated) design specifications on our Hilbert space:

1. The space should contain all HK-integrable functions.
2. The space should contain all functions in $L^2[\mathbb{R}^n]$ as a continuous dense embedding.
3. The space should contain the space of test functions as a continuous dense embedding.

To construct our space, fix n, and let $\mathbb{Q}^n$ be the set all vectors $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. This is a countable dense set in $\mathbb{R}^n$, so we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \cdots\}$. For each i and j, let $\mathbf{B}_i(\mathbf{x}_j)$ be a closed cube centered at $\mathbf{x}_j$ of side $r_i = 2^{-i}, i \in \mathbb{N}$ parallel to the coordinate axis. Now choose an order such that the set $\{\mathbf{B}_k(\mathbf{x}_k),\ k \in \mathbb{N}\}$ contains all closed cubes $\{\mathbf{B}_i(\mathbf{x}_j)\ \ |(i,j) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbf{B}_k(\mathbf{x}_k)$ and be zero otherwise, so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. We define $F_k(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}. \tag{28}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each k, $\|F_k\|_\infty \leqslant 1$ and if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. Let $t_k > 0$, $\sum_{k=1}^{\infty} t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$ as follows:

$$d\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum\nolimits_{k=1}^{\infty} t_k \mathcal{E}_k(\mathbf{x}) \mathcal{E}_k(\mathbf{y})\right] d\mathbf{x} d\mathbf{y}.$$

We can now define an inner product $(\cdot)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\mathbf{x})g(\mathbf{y})^* d\mathbf{P}(\mathbf{x},\mathbf{y}) \\ &= \sum\nolimits_{k=1}^{\infty} t_k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right]\left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y})g(\mathbf{y})d\mathbf{y}\right]^*.\end{aligned} \tag{29}$$

We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space $KS^2[\mathbb{R}^n]$. To see that it contains HK-integrable functions, let f be HK-integrable on $\mathbb{R}^n$ then:

$$\|f\|_{KS}^2 = \sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \left|\int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x}\right|^2 < \infty,$$

so $f \in KS^2[\mathbb{R}^n]$.

Definition 3.5. *We say that $\mathcal{B} \subset \mathcal{H}$ is a continuous dense embedding if:*

1. *$\mathcal{B}$ is dense in $\mathcal{H}$.*
2. *For each $f \in \mathcal{B}$, $||f||_{\mathcal{H}} \leq ||f||_{\mathcal{B}}$.*

Theorem 3.6. *For each p, $1 \leqslant p \leqslant \infty$, $L^p[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ is a continuous dense embedding.*

Proof. By construction, $KS^2[\mathbb{R}^n]$ contains $L^1[\mathbb{R}^n]$ with continuous dense embedding, therefore we only need to show that $KS^2[\mathbb{R}^n] \supset L^p[\mathbb{R}^n]$ for $p \neq 1$. If $p < \infty, L^p[\mathbb{R}^n] \cap L^1[\mathbb{R}^n]$ is dense in $L^1[\mathbb{R}^n]$ and hence dense in $KS^2[\mathbb{R}^n]$. If $f \in L^p[\mathbb{R}^n]$ then:

$$\begin{aligned}\|f\|_{KS^2} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^{\frac{2p}{p}}\right]^{1/2} \\ &\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})\,|f(\mathbf{x})|^p\,d\mathbf{x}\right)^{\frac{2}{p}}\right]^{1/2} \\ &\leqslant \sup_k \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\,|f(\mathbf{x})|^p\,d\mathbf{x}\right]^{\frac{1}{p}} \leqslant \|f\|_p.\end{aligned}$$

Hence, $f \in KS^2[\mathbb{R}^n]$ and the embedding is continuous. For $p = \infty$, we have

$$\|f\|_{KS^2} = \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^2 \right]^{1/2}$$
$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^2 \right] [ess\sup |f|]^2 \right]^{1/2} \leqslant M \|f\|_{\infty}.$$

Thus $f \in KS^2[\mathbb{R}^n]$, and $L^{\infty}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding. □

We can also construct $KS^p[\mathbb{R}^n]$ for all p by defining

$$\|f\|_{KS^p} = \begin{cases} \left\{ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^p \right\}^{1/p}, 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|, p = \infty. \end{cases}$$

It is clear that $\|\cdot\|_{KS^p}$ defines a new norm for $L^p[\mathbb{R}^n]$. If $KS^p[\mathbb{R}^n]$ is the completion of $L^p[\mathbb{R}^n]$ with respect to this norm, we have:

Theorem 3.7. *For each q, $1 \leqslant q \leqslant \infty$, $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a continuous dense embedding and, $KS^{\infty}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ for each p.*

Remark 3.8. *We note that $KS^2[\mathbb{R}^n] = KS^2[\mathbb{R}^n]^* \supset L^1[\mathbb{R}^n]^{**} = \mathfrak{M}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, and in particular, the Dirac measure $\delta(\mathbf{x}) \in \mathfrak{M}[\mathbb{R}^n]$. The theory of distributions was developed via topological vector space methods because of the belief that there is no Hilbert space that contains the test functions $\mathcal{D}[\mathbb{R}^n]$. The next result shows that their mistake was due to a lack of interest in nonabsolutely integrable functions.*

Definition 3.9. *Let $C_c^{\infty}[\mathbb{R}^n]$ be the set of infinitely differentiable functions with compact support defined on $\mathbb{R}^n$. A sequence ϕ_j is said to converge to ϕ in the sequential compact support topology, if there exists a compact set $K \subset \mathbb{R}^n$ containing the support of $\phi_j - \phi$ for all j and, for every multi-index α, $D^{\alpha}\phi_j$ converges to $D^{\alpha}\phi$ uniformly on K. With this topology we call $C_c^{\infty}[\mathbb{R}^n]$ the space of test functions and denote it by $\mathcal{D}[\mathbb{R}^n]$.*

Theorem 3.10. *The test functions $\mathcal{D}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding.*

Proof. Because $KS^{\infty}(\mathbb{R}^n)$ is continuously embedded in $KS^2[\mathbb{R}^n]$, it is sufficient to prove the result for $KS^{\infty}(\mathbb{R}^n)$. Suppose that $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, so that there exists a compact set $K \subset \mathbb{R}^n$, containing the support of $\phi_j - \phi$ and $D^{\alpha}\phi_j$ converges to $D^{\alpha}\phi$ uniformly on K for every multi-index α. Let $L = \{l \in \mathbb{N} : \text{the support of } \mathcal{E}_l,\ stp\{\mathcal{E}_l\} \cap K \neq \emptyset\}$ and $M = \sup_{l \in L}[vol(\mathbf{B}_l)]$, then

$$\lim_{j\to\infty} \|D^{\alpha}\phi - D^{\alpha}\phi_j\|_{KS^{\infty}} = \lim_{j\to\infty} \sup_{l\in L} \left| \int_{\mathbb{R}^n} [D^{\alpha}\phi(x) - D^{\alpha}\phi_j(x)]\, \mathcal{E}_l(x)\, dx \right|$$
$$\leqslant M \lim_{j\to\infty} \sup_{x\in K} |D^{\alpha}\phi(x) - D^{\alpha}\phi_j(x)| = 0.$$

Thus, as the convergence is uniform, it follows that $\mathcal{D}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding. Because $KS^2[\mathbb{R}^n]$ is self-dual, we see that the Schwartz distributions, $\mathcal{D}^*[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. □

If $f, g \in L^1[\mathbb{R}^n]$, we denote the Fourier transform of f by $\mathfrak{F}(f)$ and the convolution of g with respect to f by $\mathfrak{C}_f(g)$.

Theorem 3.11. *The Fourier transform $\mathfrak{F}(\cdot)$ and, for each $f \in L^1[\mathbb{R}^n]$, the convolution with respect to f, $\mathfrak{C}_f(\cdot)$ both extend to $KS^2[\mathbb{R}^n]$ as bounded linear operators.*

Remark 3.12. *The possibility that $KS^2[\mathbb{R}^n]$ could exist was noticed by Gill and Zachary after reading a paper by Kuelbs [24]. It was constructed by V. Steadman and first appeared as a part of her dissertation (see [42]). As shown in the next section, this space is perfect for the Feynman path integral formulation of quantum mechanics.*

3.4 Feynman Path Integral

The properties of $KS^2[\mathbb{R}^n]$ derived earlier shows that it is a better replacement of $L^2[\mathbb{R}^n]$ for the study of the path integral formulation of quantum theory developed by Feynman. It is easy to prove that both the position and momentum operators have closed, densely defined extensions to $KS^2[\mathbb{R}^n]$. Furthermore, the extension of $\mathfrak{F}(\cdot)$ and $\mathfrak{C}_f(\cdot)$ by Theorem 3.11 ensures that both Schrödinger and Heisenberg theories have faithful representations on $KS^2[\mathbb{R}^n]$.

Because $KS^2[\mathbb{R}^n]$ contains $\mathfrak{M}[\mathbb{R}^n]$ the space of all finitely and countably additive measures as compact embeddings, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $KS^2[\mathbb{R}^n]$ topology. Thus, the finitely additive set function defined on the Borel sets $B \in \mathfrak{B}[\mathbb{R}^n]$ (Feynman kernel):

$$\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B] = \int_B (2\pi i(t-s))^{-1/2} \exp\{i|\mathbf{x}-\mathbf{y}|^2/2(t-s)\} d\mathbf{y}$$

is in $KS^2[\mathbb{R}^n]$ and $\|\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B]\|_{KS^2} \leqslant 1$ and

$$\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B] = \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ \tau, d\mathbf{z}] \mathbb{K}_{\mathbf{F}}[\tau, \mathbf{z};\ s, B], \ \text{(HK-integral)}.$$

Definition 3.13. *Let* $\{\mathbf{P}_m = \{t_0^m, \tau_1^m, t_1^m, \tau_2^m, \cdots, \tau_m^m, t_m^m\}, m \leq [|\lambda t|] \ \}$ *be a family of HK-*δ_m *partitions of the interval* $[0,t]$, *where* $\mathbf{P}_k \subset \mathbf{P}_{k+1}$, *and* $\delta_{k+1}(t) \leq \delta_k(t)$, $\tau_0^m = 0$ *and for* $\psi \in KS^2[\mathbb{R}^n]$ *define: (we drop the superscript)*

$$\int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(0)] = e^{-\lambda t} \sum_{k=0}^{[|\lambda t|]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{F}}[t_j, \mathbf{x}(\tau_j);\ t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\begin{aligned} &\int_{\mathbb{R}^{[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(0)] \psi[\mathbf{x}(0)] \\ &= \lim_{\lambda \to \infty} \int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)] \psi[\mathbf{x}(0)] \end{aligned} \tag{30}$$

whenever the limit exists.

The following is elementary, but a more general result covering almost all applications recently appeared in [8].

Theorem 3.14. *The function* $\psi(\mathbf{x}) \equiv 1 \in KS^2[\mathbb{R}^n]$ *and*

$$\int_{\mathbb{R}^{3[s,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)] = \mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, \mathbf{y}] = \frac{1}{\sqrt{2\pi i(t-s)}} \exp\{i|\mathbf{x}-\mathbf{y}|^2/2(t-s)\}.$$

Remark 3.15. *The above result is what Feynman was trying to obtain. However our approach is based on a photograph of the set of bubble tracks created*

by a particle as it traverses the chamber. Each τ_i is the center of a bubble and, for each t, λt is the average time between bubbles, while the Poisson distribution gives the probability that a given photograph has n bubbles.

Because $L^2[\mathbb{R}^n]$ is dense, $KS^2[\mathbb{R}^n]$ is separable and every orthonormal basis for $L^2[\mathbb{R}^n]$ may be used to form an orthonormal basis for $KS^2[\mathbb{R}^n]$. Thus they are unitarily equivalent; however.

1. *$KS^2[\mathbb{R}^n]$ contains both $\mathcal{D}[\mathbb{R}^n]$ and $\mathcal{D}^*[\mathbb{R}^n]$, whereas $L^2[\mathbb{R}^n]$ does not.*
2. *The integral of e^{ix^2} is finite in $KS^2[\mathbb{R}^n]$ but infinite in $L^2[\mathbb{R}^n]$.*

Thus, $L^2[\mathbb{R}^n]$ and $KS^2[\mathbb{R}^n]$ are mathematically equivalent but not physically equivalent.

4 Horwitz-Fanchi Historical Time

Newton suggested that the universe had an absolute definition of time that existed independently of any observer. He assumes that this clock provides a unique definition of simultaneity and ordering for events, but may only be understood as a mathematical object. (In Newton's era mathematics and personal philosophy still played a dominant role in physics). Einstein's special theory became problematic for Newton's absolute time, whereas Minkowski's postulate eliminated any sense of ordering. Feynman later suggested that we view a physical event as a 3-dimensional motion picture, exposing more of the outcome as the film unfolds (reintroduced the ordering property).

The first use of a fifth parameter to account for the missing order property was Fock, and later Stueckelberg, followed by Feynman and Schwinger. They each considered it a global clock in addition to the Minkowski space-time. Horwitz and Piron [21], and Fanchi and Collins gave it the name historical time. They were the first to realize that there should be physical justification for this clock (see [11]).

The purpose of this section is to prove this conjecture. Our proof uses the third definition of proper time and is based on a relatively weak assumption regarding the properties of the universe.

Theorem 4.1. *If the universe is representable in the sense that it is independent of any observer's particular portion of the universe, the universe has a unique clock.*

Proof. If Mc^2 is the total mass energy and H is the total energy Hamiltonian for the universe as seen by O, under the stated conditions H/Mc^2 is constant. By assumption this view is observer independent, so that O' will also obtain the same ratio. It follows from (3) that

$$d\tau = \frac{Mc^2}{H}dt = \frac{Mc^2}{H}dt', \quad \Rightarrow \int \tfrac{Mc^2}{H}dt = \int \tfrac{Mc^2}{H}dt' \equiv \tau_h$$

and τ_h uniquely defines a clock for the universe. □

The principle of homogeneity states that the distribution of matter does not vary significantly with its position on a large scale. Several recent studies have suggested that the principle of homogeneity is not as secure as believed in the past (see [30]). Recent studies have also led to questions about the principle of isotropy (see [39]). Thus, if future research supports these results, the cosmological principle may not hold true. However, the assumption of Theorem 4.1 can still remain valid.

4.1 The Feynman Interpretation of Antiparticles

One might naturally suspect that Dirac would have noticed that the positron could be interpreted as an electron moving backward in time after developing his hole theory. However, this idea was first developed by Feynman following the suggestion of Wheeler a few years before Stueckelberg introduced the same idea (for details, see Schweber [45]). However, the correct mathematical justifcation for such an interpretation only became possible in 1980's after the invention of dual numbers by Santilli (see [SN]). In this section, we show how use of these numbers makes Feynman's approach natural.

Formally, the real numbers is a quadruple $(\mathbb{R}, +, \cdot, \rightarrow)$, with 0 as the additive identity and 1 as the multiplicative identity. We have included $\rightarrow$ to include an unacknowledged sense of direction. If $\hat{\mathbb{R}} = \{-a : a \in \mathbb{R}\}$, then:

Definition 4.2. *The dual of the real numbers* $(\hat{\mathbb{R}}, +, * \leftarrow)$ *is a field, with* $0 = \hat{0}$ *as the additive identity,* $-1 = \hat{1}$ *as the multiplicative identity, and*

$$\hat{a} * \hat{b} = (-a)(-1)(-b) = (-ab) = \hat{ab} \text{ for all } \hat{a}, \hat{b} \in \hat{\mathbb{R}}.$$

Example 4.3. *The following example for quantum theory is as follows: the proper time evolution of a particle is defined on a Hilbert space* $\mathcal{H}$ *over complex numbers* $\mathbb{C} = \mathbb{R} + i\mathbb{R}$*, with Hamiltonian* K *and is given by the equation:*

$$i\hbar\frac{\partial\psi}{\partial\tau} = K\psi.$$

The conjugate equation is:

$$-i\hbar\frac{\partial\psi^*}{\partial\tau} = K\psi^*.$$

If we use $\hat{\mathbb{C}}$ as our number field, we can write the above equation as:

$$\hat{i} * \hat{\hbar} * \frac{\partial\hat{\psi}^*}{\partial\hat{\tau}} = \hat{K} * \hat{\psi}^*$$

This correctly defines an antiparticle as a particle with its proper time reversed, and with its evolution defined on $\mathcal{H}^*$ over $\hat{\mathbb{C}}$.

Remark 4.4. *It has been shown by Santilli that charge conjugation and isoduallity (as he calls it) are both equivalent for the particle-antiparticle symmetry operation.*

Discussion and Conclusion

In this study we have faced a number of foundational mathematical problems that have clouded our ability to correctly address some of the physical problems left over from the last century.

4.2 Classical Level

At the classical level, we have made Minkowski's unacknowledged third (mathematical) postulate for Einstein's special theory of relativity explicit, and provided direct proof that it is incompatible with the first postulate for two or more particles. We then used one of the three representations for the proper time to show how to resolve this incompatibility, allowing us to resolve the center of mass problem noticed by Pryce [37], the no Interaction problem first conjectured by Bakamjian and Thomas [2], and later proved by Currie, Jordan and Sudarshan [4]. This clarifies the way for a direct extension of Newton's theory to one compatible with that of Maxwell (addressed in the second paper).

4.3 Quantum Level

We first show that unconstrained by the Minkowski postulate, a new KG-equation is possible, which is valid for all spins. We then provide an analytical separation (diagonalization) of the Dirac equation into particle and

antiparticle components. The diagonalization is analytic in that it is achieved without transforming the wave functions and reveals the time nonlocal behavior of the particle-antiparticle relationship. The probability densities are also derived giving a complete decoupling of the spinor space $L^2[\mathbb{R}^3, \mathbb{C}^4] = L^2[\mathbb{R}^3, \mathbb{C}^2] \oplus L^2[\mathbb{R}^3, \mathbb{C}^2]$. We have then shown that the square root operator is nonlocal in space but related by a unitary transformation, implying that the mathematical definition of equivalence need not be physical.

4.3.1 Pauli Approximation

It is generally agreed that QED is in excellent agreement with experimental results. The fact that it is very successful is not in doubt. However, there was still some serious technical and foundational issues that need to be addressed. Historically, when Lamb and Retherford confirmed that the $2s_{1/2}$ state of hydrogen was shifted above the $2p_{1/2}$ state, and the Pauli approximation to the Dirac equation was used to determined that the Dirac equation was not sufficient. Thus, from a foundational perspective, it is important to understand the extent to which physical statements about the s-state behavior of the Dirac equation based on the Pauli equation (or any of its variants) are correct. We first recall that a Dirac particle has a finite probability of being at the origin in the s-state. This is not true for p-states or higher, so the Pauli approximation is justified. However, we explicitly show that the Pauli approximation is not valid for any calculation in the s-states, for two reasons.

1. We cannot replace $(E - V + mc^2)$ with $2mc^2$ because V is undefined at $r = 0$.

2. The correct probability density for the Pauli approximation is $\rho_\psi = |\psi(\mathbf{x})|^2 + \left|\frac{(\sigma\cdot\pi)}{2mc}\psi(\mathbf{x})\right|^2$ but it cannot be used to compute s-state perturbations because it contains an r^{-2} term making it undefined at $r = 0$.

We then show that these difficulties can be avoided, with an error of $\approx 10^{-5}$eV, provided a cut-off is imposed. An important conclusion is that a cut-off is required before QED. This implies that extra covariant formulations and renormalization procedures may not be necessary to obtain and justify the excellent QED results.

4.4 Feynman Path Integral

Historically, the mathematics community has responded in a number of ways to the introduction of new mathematical ideas or methods from the outside. The first response is to fit it into an existing framework. The second and most exciting response is when such an idea or method leads to the development of a new branch of mathematics. The most prevalent and successful response has been to find an existing mathematical structure that reasonably accommodates the physical theory and provides (at least) a framework for mathematical rigor. In some rare but important instances, no obvious mathematical structure can completely accommodate the theory in the manner presented by physicists. In this case, mathematicians have extended and/or adapted an existing mathematical theory, developed new mathematical structures or suggested (in frustration) that any conclusions derived from the use of these ideas or methods are suspect. Over the last 70 years, all of the above positions have appeared in response to to the introduction of Feynman's path integral into quantum theory.

In this study, we have introduced a more general integral (HK) and designed a new Hilbert space ($\mathcal{KS}^2[\mathbb{R}^n]$) for the Feynman theory. This approach provides the correct framework for mathematical rigor without removing the physically intuitive and computationally effective approach suggested by Feynman. This space is also unitarily equivalent to $L^2[\mathbb{R}^n]$ showing that unitary equivalence is not always indicative of the same physics.

4.5 Feynman's Interpretation of Antiparticles.

In the last section, we introduced the dual of the real numbers and used them to provide mathematical meaning for Feynman's idea of anti particles with their proper time reversed.

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Declaration

The authors certify that:

1. There is no data developed or attached to this paper.
2. They have no relevant financial or non-financial interests to disclose.
3. There are no conflicts of interest to declare that are relevant to the content of this manuscript.
4. The authors have no financial or proprietary interests in any material discussed in this manuscript.
5. They have no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript.

References

[1] H.R Brown, Eur. J. Phys. **26**, S85 (2005).

[2] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[3] K.M Case, Phys. Rev. **111**, (1958) 1011.

[4] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).

[5] P. A. M. Dirac, Proc. Roy. Soc (London) **A117** (1928) 610, **A118** (1928) 351.

[6] A. Einstein, Ann. d. Phys. **17**, 891 (1905).

[7] A. Einstein, Jahrbuch Radioaktivitat **V**, 422 (1907) (Berichtigungen).

[8] G. Esposito and T. L. Gill, *On the physical and mathematical foundations of quantum physics via functional Integrals*, (to appear) Int. J. Mod. Phys. (2025) 2530011.

[9] A. Einstein and J Laub, Über die elektromagnetischen Grundgleichugen für bewegte Köper, *Ann. d. Phys.* **26** (1908), 532-540.

[10] A. Einstein and J Laub, Über die im elektromagnetischen Felde auf ruhende Köper ausgeübten ponderomotorischen Kräfte, *Ann. d. Phys.*, **26** (1908) 541-550.

[11] J. R. Fanchi, *Confronting the ENIGMA of TIME*, World Scientific, Singapore, (2023).

[12] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[13] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[14] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer, N. Y., (1987).

[15] I. S. Gradshteyn and I. M. Ryzhik T̂ables of Integrals, Series and Products, Academic Press, NewYork, (1980).

[16] T. L. Gill and W. W. Zachary, *Foundations for relativistic quantum theory I: Feynman's operator calculus and the Dyson conjectures*, Journal of Mathematical Physics **43** (2002), 69-93.

[17] T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman operator Calculus*, Springer New York, (2016).

[18] T.L. Gill, W.W. Zachary, *Analytic representation of the square-root operator*, J. Phys. A: Math. Gen. **38** (2005) 1-18.

[19] T.L. Gill, W.W. Zachary, and M. Alfred, *Analytic representation of the Dirac equation*, J. Phys. A: Math. Gen. **38** (2005) 1-22.

[20] D. Hestenes, *Space Time Algebra*, Gordon and Breach Pub. New York, (1966)

[21] L. P. Horwitz and C. Piron, Helv. Phys. Acta **46**, 316 (1981).

[22] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, (1991).

[23] L. P. Horwitz, *On the significance of a recent experiment demonstrating quantum interference in time*, (to appear in Phys. Rev. Letters, see arXiv:quant-ph/0507044).

[24] J. Kuelbs, *Gaussian measures on a Banach space,* Journal of Functional Analysis **5** (1970), 354–367.

[25] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zür Mathematik, Band **26**, Teubner Verlagsgesellschaft, Leipzig, (1980).

[26] H. A. Lorentz, *Archives Nerlandaises des Sciences Exactes Naturelles*, **25**, (1892).

[27] H. A. Lorentz, *The Theory of Electrons* B. G. Teubner, Leipzig, 1906; (reprinted by Dover, New York, 1952).

[28] H. Leutwyler. *A no-interaction theorem in classical relativistic hamiltonian particle mechanics*, Nuovo Cim., **37**, 556 (1965).

[29] G. Longhi, L. Lusanna, and J. M. Pons, J. Math. Phys. 30, 1893 (1989).

[30] Nathan J. Secrest; Sebastian von Hausegger; Mohamed Rameez; Roya Mohayaee; Subir Sarkar; Jacques Colin. *A Test of the Cosmological Principle with Quasars*, The Astrophysical Journal Letters, **908** (2) 2021): L51

[31] A. Pais, *Subtle is the lord: the science and life of Albert Einstein*, Oxford University Press (1982).

[32] W. Perret and G. B. Jeffery (translators, with additional notes by A. Sommerfeld), *The Principle of Relativity* by H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, Dover, New York, 1952.

[33] H. Poincaré, *Sur la dynamique de l' électron*, Rendiconti del Circolo matematico Rendiconti del Circolo di Palermo, **21**, 129-176 (1906).

[34] D. L. Pursey, Nucl. Phys. **53**, (1964) 174.

[35] W. Pauli, *Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren.* Zeitschrift für Physik. **31** (1): 765-783 (1925).

[36] W. Pauli, Handbuch der Physik 2nd ed. **24**, (1933).

[37] M.H.L. Pryce, Proc. Roy. Soc. London A **195**, 400 (1948).

[38] S. G. Samko, *Hypersingular Integrals and Their Applications*, Analytical Methods and Special Functions, Vol. **5** Taylor & Frances, Pub., (2002).

[39] Saadeh D, Feeney SM, Pontzen A, Peiris HV, McEwen, JD . *How Isotropic is the Universe*, Physical Review Letters. **117** (13) (2016) 131302.

[40] L. I. Schiff, 1955 *Quantum Mechanics*, McGraw-Hill, New York (1955).

[41] J. C. Slater, *Quantum Theory of Atomic Structure*, Vol. II McGraw-Hill, New York (1960).

[SN] R. M. Santilli, *Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isoduals, and pseudoduals, and hidden numbers of dimension 3, 5, 6, 7*, Algebras, Groups and Geometries, Vol. **10**, 273?322 (1993).

[42] V. Steadman, *Theory of operators on Banach spaces*, Ph.D thesis, Howard University, 1988.

[43] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, New York, (1964).

[44] S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson & Co., New York, (1961).

[45] S. Schweber, *QED And The Men Who Made It* , Princeton University Press, London (1994).

[46] S. Walters, In: Gonner, H., Renn, J., Ritter, J. (eds.) *The Expanding Worlds of General Relativity.* Einstein Studies, vol. 7, pp. 45-86. Birkhäuser, Boston (1999).

[47] A. O. Williams, Phys. Rev. **58** (1940), 723.

[48] S. Weinberg, *High energy behavior in quantum field theory,* Phys. Rev. **118** (1960), 838-849.

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HAWKING PROCESS AND NEGATIVE ENERGY

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Abstract

Hawking radiation provides a possible mechanism for the production of negative energy. While this negative energy accumulates inside the event horizon, positive energy is radiated from the black hole. The negative energy facilitates a number of physical effects that foster the stability of black holes and wormholes. Negative energy was previously investigated as a consequence of traditionally excluded Feynman diagrams. These diagrams provide a possible theoretical mechanism for the creation of negative energy. An evaluation of the associated amplitudes of these diagrams suggest that light field particles and photons are the dominant component of the Hawking radiation since their amplitudes scale as the reciprocal of the created particle mass.

KEYWORDS: Hawking radiation; black holes; Schwarzschild metric; negative energy creation; and black hole thermodynamics.

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1.0 Introduction

A number of papers address Hawking radiation, associated particle production, and black hole thermodynamics [1-12]. As a prelude to Hawking's radiation assertion, Bekensteing [1] provided essential justification and discussion of a number of similarities between black-hole physics and thermodynamics. Prior to the radiation assertion, Hawking [2] showed that a black hole creates and emits particles (e.g., neutrinos and photons) at a rate consistent with a body having a temperature proportional to the surface gravity of the black hole. Accordingly, as a black hole emits this thermal radiation, it loses mass.

Hawking noted that black holes are not static, but continuously radiate energy [3]. This situation arises from quantum effects in the electromagnetic field of a black hole. According to the uncertainty principle, electromagnetic field particles or photons are not precisely localized. In the vicinity of the black hole horizon, Hawking utilized quantum field theory to describe the properties of photons near the event horizon [3]. Page provided a comprehensive review of Hawking radiation and black hole thermodynamics [4].

As an extension of Hawking's work, Firouzjaee, and Ellis [5] addressed the black hole particle creation probability rate resulting from quantum fluctuations. [5] suggests that the Hawking radiation emission process also occurs for a dynamic black hole when it has a slowly evolving horizon.

Steinhauer [6] noted spontaneous Hawking radiation, stimulated by quantum vacuum fluctuations, resulted from an analogue black hole in an atomic Bose–Einstein condensate. These observations suggest an approximately thermal distribution of Hawking radiation.

Other aspects associated with Hawking radiation and black hole thermodynamics are noted in [7 – 12]. For example, Kolobov et al. [12] noted that Hawking radiation from a black hole is a slowly evolving process, and this condition is required for the correspondence between Hawking and blackbody radiation.

Additional aspects of related cosmology [13-14] and conventional thermodynamics and its applications are provided by Santilli [15-16]. These papers provide supporting commentary regarding the thermodynamic and cosmological basis for the Hawking radiation process.

These various aspects of Hawking radiation and associated black hole thermodynamics are contingent on particle creation near the event horizon. The implications of the creation of negative energy particles [17,18] near the event

horizon, and the connection to traditionally excluded Feynman diagrams are the subject of this paper.

2.0 Theoretical Description of Hawking Radiation

Hawking derived the black hole radiation concept utilizing a rigorous quantum field theory approach [3]. The Hawking radiation concept can also be derived in a somewhat simplified form by considering the implications of the uncertainty relationship [19]

$$\Delta E\, \Delta t \geq \hbar/2 \qquad (1)$$

where ΔE is the minimum uncertainty in the created particle energy that resides in a quantum state for a time Δt. As noted in previous publications [17-18], spacetime allows vacuum fluctuations in the electromagnetic fields that produce pairs of particles including photons that are produced at one event location and destroyed at another. The created pairs violate conservation of energy by an amount ΔE unless they exist for less than a time $\Delta t = \hbar/2\Delta E$. In that event, no conservation law is violated. Consistent with quantum field theory, on shell conservation of energy is achieved (at a macroscopic scale), while it is violated off shell (on a quantum level).

Spacetime near the event horizon of a black hole is locally flat, and quantum fluctuations will routinely occur. If a fluctuation produces a photon pair, one has energy E and the other -E. In flat spacetime, the negative energy photons combine with the positive energy ones before the time $\hbar/2E$ elapses. However, if the fluctuation produces a pair just outside the event horizon, a negative energy photon could cross the horizon before the time $\hbar/2E$ elapses. Once inside the black hole, the negative energy photon can move freely.

The following discussion is consistent with the more comprehensive treatment of Schultz [19]. Readers should consult [19] for additional details supporting the Section 2.0 commentary.

Considering the Schwarzschild metric, negative energy is normally excluded because it corresponds to a particle that travels backward in time [19]. Inside the event horizon, an observer going forward in time is traveling in the direction of decreasing r. As an example, select a trajectory characterized by $p^0 = 0$ and $U^0 = 0$, where U is the four velocity and p is the four momentum. Given these conditions and assuming only radial motion, U^r is the only nonzero four-velocity component. The usual normalization condition applies,

$$U \cdot U = -1 \tag{2}$$

and yields the value U^r for the Schwarzschild metric

$$U^r = -\left(\frac{2M}{r} - 1\right)^{1/2} \qquad \text{for } r < 2M \tag{3}$$

where M is the mass in the Schwarzschild metric. U^r is negative because the observer is ingoing. For the condition $-p \cdot U > 0$, any photon orbit is allowed.

Given this condition, consider a zero angular momentum photon that moves radially inside the event horizon. As noted in [19], this photon has energy $E = \pm p^r$, and its energy relative to the observer is

$$-p \cdot U = -g_{rr}\, p^r\, U^r = -\left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}\, p^r \tag{4}$$

where $g_{rr} = \left(\frac{2M}{r} - 1\right)^{-1}$ for the Schwarzschild metric. This is positive if the photon trajectory is ingoing ($p^r < 0$). However, the aforementioned condition establishes no restrictions on the energy E. Photons can follow null trajectories inside the horizon and can have either sign for E as long as $p^r < 0$.

A fluctuation near the event horizon can create a trajectory in which the positive energy photons move radially outward from the horizon, and can escape to infinity. The negative energy photons can travel inward and remain inside the event horizon.

For this condition, consider the fluctuations in a freely falling inertial reference frame (i.e., one that is locally flat). A frame that is momentarily at rest at the coordinate $r = 2M + \varepsilon$ will fall inward following the trajectory of a particle with zero angular momentum and energy ($\bar{E}$) [19]

$$\bar{E} = \left[1 - \frac{2M}{2M+\varepsilon}\right]^{1/2} \approx \left(\frac{\varepsilon}{2M}\right)^{1/2} \tag{5}$$

The photon reaches the event horizon after a proper time $\Delta\tau$

$$\Delta\tau = -\int_{2M+\varepsilon}^{2M} \left[\frac{2M}{r} - \frac{2M}{2M+\varepsilon}\right]^{-1/2} dr \tag{6}$$

To first order in ε, $\Delta\tau$ is

$$\Delta\tau = 2\,(2M\epsilon)^{1/2} \tag{7}$$

The photon energy Σ in this frame is obtained by setting $\Delta\tau$ equal to the fluctuation time $\hbar/2\Sigma$ derived from the uncertainty relationship

$$\Sigma = \frac{\hbar}{4}\,(2M\varepsilon)^{-1/2} \tag{8}$$

In (8), Σ is the energy of the outgoing photon (i.e., the one that reaches infinity) as determined in the local inertial reference frame. The energy of this photon when it reaches infinity is [19]

$$\Sigma = -\vec{p}\cdot\vec{U} \tag{9}$$

where

$$-U_0 = \bar{E} \approx \left(\frac{\varepsilon}{2M}\right)^{1/2} \tag{10}$$

Therefore,

$$\Sigma = -g^{00}\,p_0\,U_0 = g^{00}\,U_0\,E \tag{11}$$

where E is the conserved energy on the photon's trajectory, and is also the energy when the outgoing photon reaches infinity. When g^{00} is evaluated at $2M + \varepsilon$,

$$E = \Sigma\left(\frac{\epsilon}{2M}\right)^{1/2} = \frac{\hbar}{8M} \tag{12}$$

This is the characteristic energy of the photon, independent of its origin, when it escapes from the black hole.

Hawking radiation is significant since it applies to several geometries (i.e., Kerr (rotating) and Schwarzschild (static) black holes). In addition, it connects the negative energy photons inside the event horizon with their positive energy counterparts outside the horizon. Hawking radiation is steady thermal energy created by quantum fluctuations near the horizon.

This result suggest that as positive energy is radiated from the black hole, negative energy accumulates inside the event horizon. The possible consequences of this accumulation of negative energy was noted in previous publications [17,18].

3.0 Implications of Negative Energy

In previous papers [17-18], negative energy was noted to theoretically arise as a consequence of forbidden S2 (Figs, 1 and 2) as well as coupled $S^1 \times S^1$ (Fig, 3) Feynman diagrams. These initial assessments of a Planck-Scale quantum field theory identified three basic Feynman diagrams that could generate negative energies (i.e., vacuum bubbles, closed loops, and coupled single vertex diagrams at the Planck-scale). These diagrams are discussed in detail in Refs, [17-18], and a summary is provided in subsequent commentary.

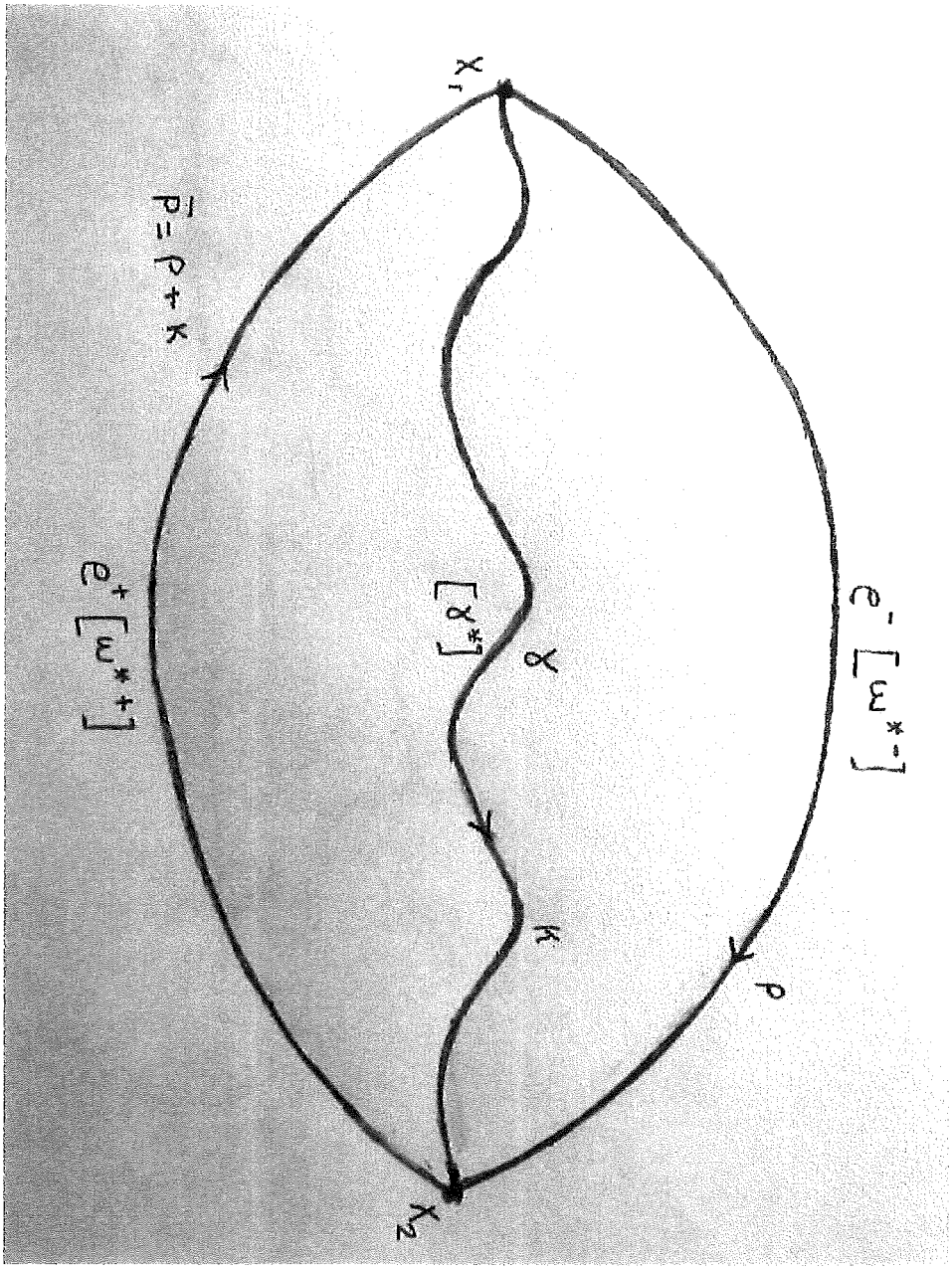

Fig. 1. Vacuum bubble Feynman diagram and the generalized Planck-scale analogue. The S_2 vacuum bubble produces negative energy particles. The terms in brackets refer to particles arising at the Planck-scale. Other quantities (electrons, positrons, and photons) refer to standard QED processes. The two processes illustrated in the figure are separate and distinct with the "*" superscript denoting Planck-scale particles.

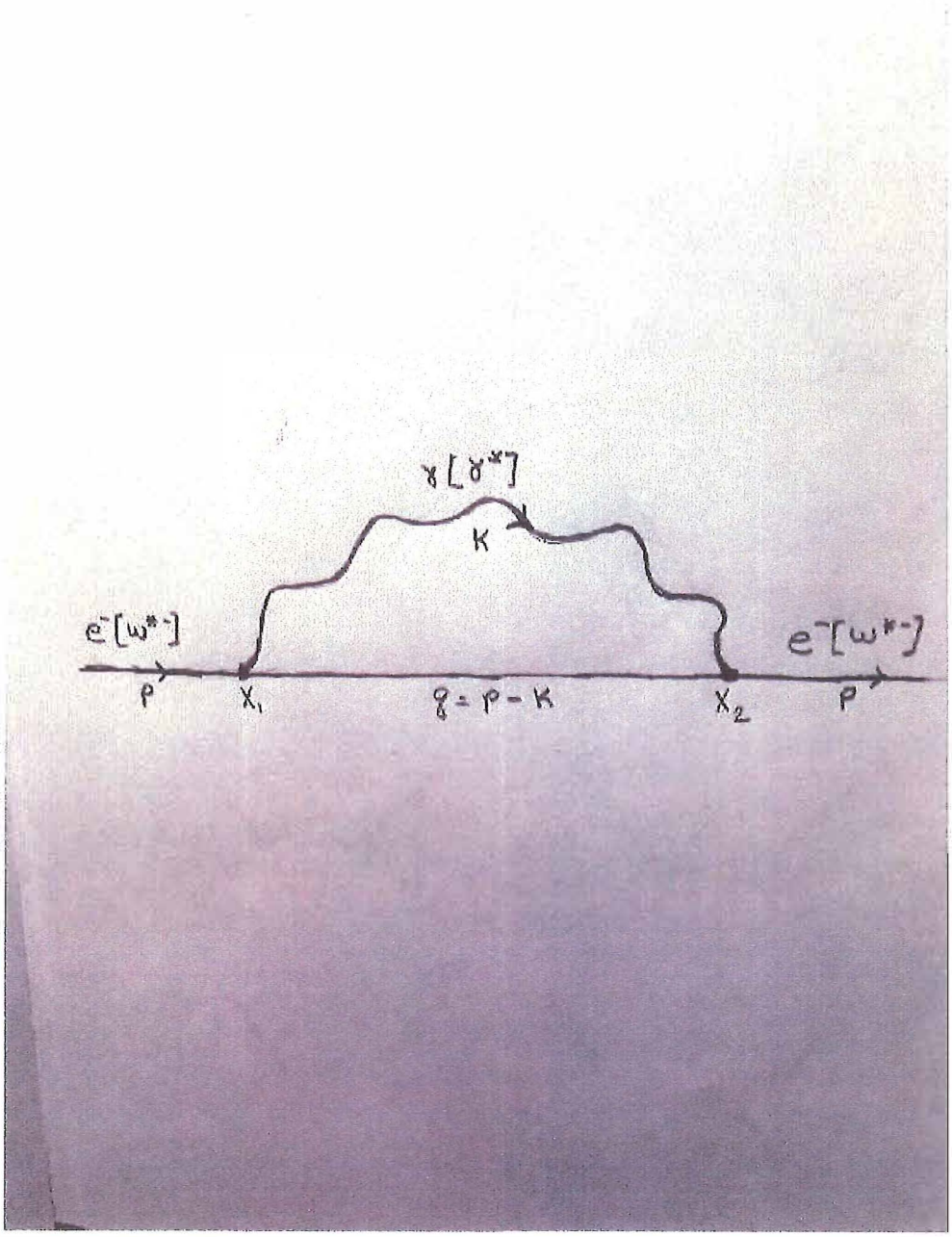

Fig. 2. Closed loop Feynman diagram and the generalized Planck-scale analogue. The S_2 loop diagram produces negative energy particles. The terms in brackets refer to particles arising at the Planck-scale. Other quantities (electrons and photons) refer to standard QED processes. The two processes illustrated in the figure are separate and distinct with the "*" superscript denoting Planck-scale particles.

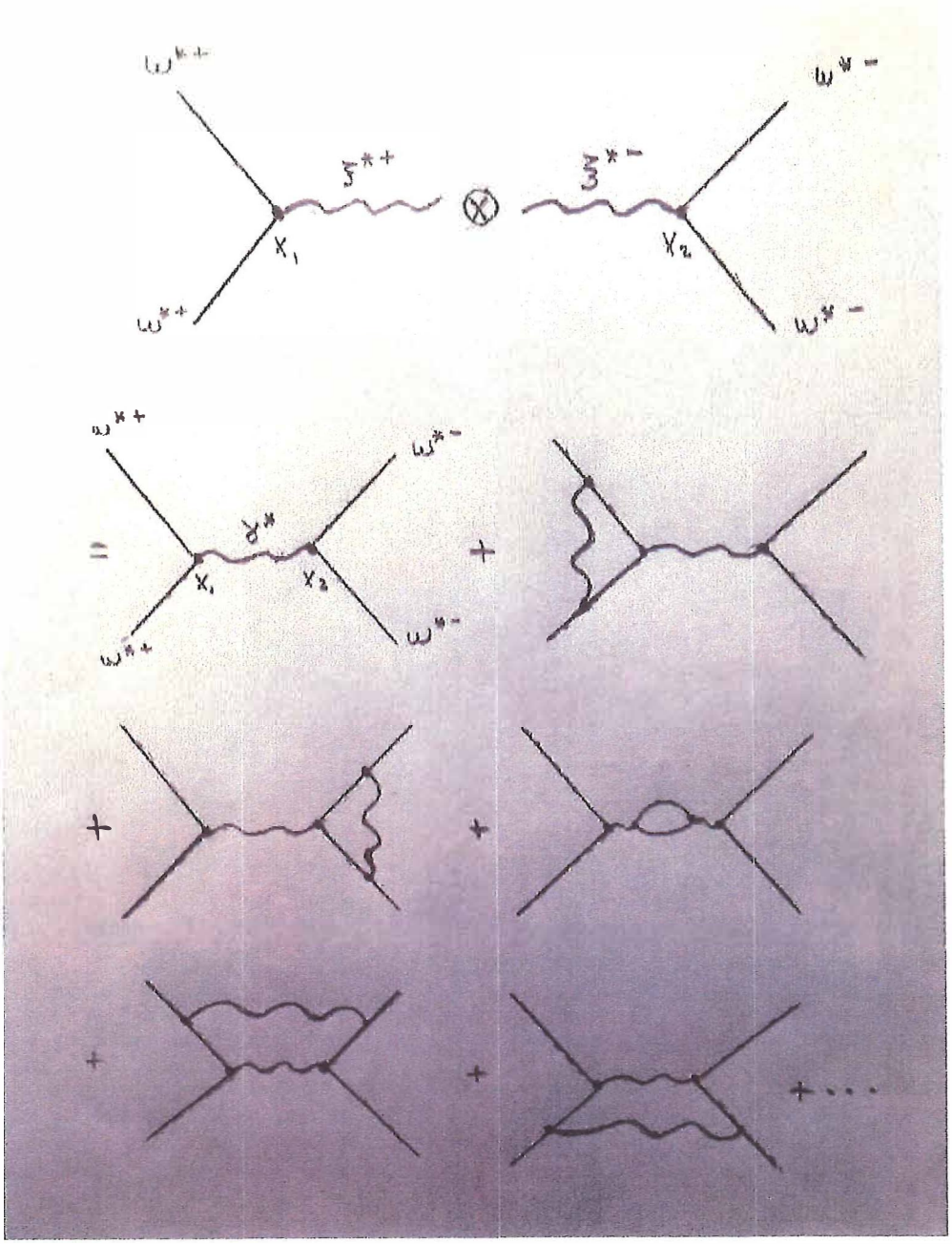

Fig. 3. Two coupled S_1 diagrams that produce positive and negative energy particles. These particles occur at the Planck-scale. The tree level and selected higher order terms are illustrated.

These diagrams were constructed in general without a specific physical context, but the previous discussion suggests that the requisite negative energy can theoretically occur through a vacuum fluctuation near the event horizon of a black hole. Previously, the origin of the negative energy was unspecified [17,18], but the region just outside the event horizon provides a possible physical creation location for the negative energy particles that are part of the Hawking radiation creation process. In addition, these negative energies were previously noted to facilitate a variety of effects including wormhole stability, black hole stability, and creation of new exotic particles [17.18]. As noted in [17-18], these particles could be created from a new class of $S^1 \times S^1$ Feynman diagrams.

The derivation of the resulting amplitudes corresponding to Figs. 1, 2, and 3 were provided in [18] and the results are summarized in subsequent discussion. In evaluating the amplitude of Fig. 1, the following assumptions were utilized: (1) basic Feynman rules are applicable, (2) two-dimensional φ^4 scalar field theory is applicable, (3) light front coordinates are utilized, and (4) QED or an analogous QFT is applicable.

Fig. 1 corresponds to a vacuum bubble with associated amplitude [18]

$$A(\text{Fig.1}) = -\frac{i\lambda^2}{\mu^2}\pi^2 NC \propto \frac{\lambda^2}{\mu^2} \tag{13}$$

where λ is the vertex coupling constant, p and k are the light front momenta defined in Fig. 1, and G(q) and N are defined as

$$G(q) = \frac{i}{q^2 - \mu^2 + i\varepsilon} \tag{14}$$

$$N = \frac{1}{3!}\frac{1}{i(2\pi)^4} \tag{15}$$

In (13-15), μ is the scalar field particle mass, the 1 / 3! is a symmetry factor, and C is a constant (~ 2.3439). The amplitude of Fig. 1 is inversely proportional to the square of the scalar field particle mass.

The amplitude of Fig. 2 corresponds to a closed loop diagram. In [18] it was evaluated assuming that (1) the basic Feynman rules are applicable, (2) two-

dimensional φ^4 scalar field theory is applicable, (3) light front coordinates are utilized, (4) QED or an analogous QFT is applicable, (5) the requisite integrals are evaluated with the cutoff limit $\Lambda \rightarrow \infty$, and (6) the amplitude is evaluated with the $p = 0$ limitation. Given these assumptions, the amplitude of Fig. 2 is

$$A(\text{Fig. 2}) = \left(\frac{i}{16\pi^2}\right)\frac{\lambda^2}{\mu^2}(-2\pi i)\int_{-\infty}^{+\infty} dk^+ \delta(k^+) = \frac{1}{8\pi}\frac{\lambda^2}{\mu^2} \propto \frac{\lambda^2}{\mu^2} \qquad (16)$$

The Fig. 2 amplitude is similar to the Fig. 1 amplitude since both results are proportional to λ^2 / μ^2. This result implies that the amplitudes for Figs. 1 and 2 are governed by the coupling constant λ and the scalar field particle mass μ.

The amplitude of Fig. 3 corresponds to a coupled diagram. In [18] it was evaluated assuming that (1) the basic Feynman rules are applicable, (2) QED or an analogous QFT is applicable, (3) The $m(\gamma^*) >> (p_1 + p_2)^2$ assumption is similar to the Fig. 2 $p = 0$ limitation, and (4) ξ^{*+} and ξ^{*-} annihilate to produce a photon analogue γ^+. For the Hawking radiation process the γ^+ is either a light scalar field particle or photon. Given these assumptions, the amplitude of Fig. 3 is similar to the previous results for the amplitudes for Figs. 1 and 2.

$$A(\text{Fig. 3}) \propto \frac{g^2}{m(\gamma)^2} \rightarrow \frac{\lambda^2}{\mu^2} \qquad (17)$$

where the substitution $g \rightarrow \lambda$ and $m(\gamma) \rightarrow \mu$ is used to facilitate comparison to the Figs. 1 and 2 amplitude results. For all three diagrams, the largest amplitudes are attributed to the smallest field particle masses. This result provides possible insight into the physical characteristics of Hawking radiation, and its relationship to Feynman diagrams that lead to the production of negative energy.

4.0 Emergence of Light Particles

The Fig. 3 Feynman diagram naturally leads to the creation of the Hawking energy photons through the coupled $S^1 \times S^1$ diagrams. Figure 3 implicitly contains two field particles that represent creation and annihilation mechanisms that could create a photon pair near the event horizon. An evaluation of the tree-level coupled diagram and other negative energy diagrams by a number of methods [18], leads to the natural emergence of amplitudes that are enhanced by the creation of low mass field particles or photons.

In [18], the S^1 x S^1 diagrams of Fig. 3 were developed as a theoretical construct to create negative energy photons. The Hawking radiation process provides a context for the possible applicability of these coupled diagrams. The creation of photons derived from these diagrams provides a possible theoretical mechanism for the Hawking radiation process. Therefore, there is a possible relationship between the S^1 x S^1 diagrams, the creation of negative energy particles or photons, and Hawking radiation. This is an important point that possibly links the concept of negative energy, Hawking photons, and unique Feynman diagrams.

5.0 Conclusions

The Hawking radiation process is an important theoretical concept that affects the behavior of black holes. Vacuum fluctuations can create photon pairs. If these pairs are created in proximity to the event horizon, the positive energy photon can escape to infinity while the negative energy photon can travel inside the event horizon and move freely.

The postulated S^1 x S^1 diagrams provide is a theoretical means for the creation of negative energy, and subsequent emission of Hawking radiation. This is an important point that possibly links the concept of negative energy, Hawking radiations, and unique Feynman diagrams.

References

[1] J. D. Bekensteing, Black Holes and Entropy, Phys. Rev. D 7, 2333-2346 (1973).
[2] S. W. Hawking, Black hole explosions?. *Nature* **248**, 30–31 (1974).
[3] S. W. Hawking, Particle Creation and Black Holes, Commun. Math. Phys. **43**, 199-220 (1975).

[4] D. N. Page, Hawking radiation and black hole thermodynamics, New Journal of Physics **7** (2005) 203, 1-33.
[5] J. T. Firouzjaee, and G. F. R. Ellis, Particle creation rate for dynamical black holes. *Eur. Phys. J. C* **76**, 620 (2016). https://doi.org/10.1140/epjc/s10052-016-4479-9.
[6] J. Steinhauer, Observation of quantum Hawking radiation and its entanglement in an analogue black hole. *Nature Phys* **12**, 959–965 (2016).
[7] G. Koczan, Derivation of Hawking Radiation Part I. Gravity and Thermodynamics of Black Holes, English translation: FOTON **140**, Wiosna 2018, 1 – 18.

[8] J. M. Dudley and D. Skryabin, New horizons for Hawking radiation, Physics 3, 95 (2010). DOI: 10.1103/Physics.3.95, URL: http://link.aps.org/doi/10.1103/Physics.3.95.

[9] A. Almheiri et. al, The entropy of Hawking radiation, Rev. Mod. Phys. **93**, (2021) 035002-1 - 035002-23.

[10] A. Arbey et. al, Hawking radiation by spherically-symmetric static black holes for all spins: Teukolsky equations and potentials, Phys. Rev. D **103**, 104010-1 - 104010-17 (2021).

[11] A. Arbey et. al, Hawking radiation by spherically-symmetric static black holes for all spins II. Numerical emission rates, analytical limits, and new constraints, Phys. Rev. D **104**, 084016-1 - 084016-19 (2021).

[12] Kolobov, V.I., Golubkov, K., Muñoz de Nova, J.R. et al. Observation of stationary spontaneous Hawking radiation and the time evolution of an analogue black hole. Nat. Phys. **17**, 362–367 (2021).

[13] R. M. Santilli, A new cosmological conception of the universe based on the isominkowskian geometry and its isodual, Contributed paper in Analysis, Geometry and Groups, A Riemann Legacy Volume, Volume II, H.M. Srivastava, Editor, pp. 539-612 (1993).

[14] R. M. Santilli, A new cosmological conception of the Universe based on the isoriemannian geometry and its isodual, in Analysis, Geometry and Groups: A Riemann Legacy Volume, Part II, H. M. Srivastava and Th.M. Rassias, Editors Hadronic Press, pp. 539 (1994).

[15] R. M. Santilli, HADRONIC MATHEMATICS, MECHANICS AND CHEMISTRY, Volume I: Limitations of Einstein's Special and General Relativities, Quantum Mechanics and Quantum Chemistry, Institute for Basic Research, Palm Harbor, FL (2008).

[16] R. M. Santilli, HADRONIC MATHEMATICS, MECHANICS AND CHEMISTRY, Volume II: Isodual Theory of Antimatter Antigravity and Spacetime Machines, Institute for Basic Research, Palm Harbor, FL (2008).

[17] J. J. Bevelacqua, Possible Planck-scale Physical Production Mechanisms and Consequences of Negative Energies – Initial Formulation, Physics Essays **34**(3), 342 (2021).

[18] J. J. Bevelacqua, Planck-Scale Quantum Field Theory –II- Evaluation of QED Analogue Amplitudes, Hadronic Journal **45**, 235 (2022).

[19] B. Schultz, A First Course in General Relativity, Second Edition, Cambridge University Press, Cambridge, UK (2016).

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DOI 10.29083/HJ.48.04.2025/SC439

EXPANSION FORCE AS A BOHMIAN QUANTUM EFFECT TO TEST HUBBLE CONTROVERSY AND EARLY DARK ENERGY

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Abstract

An additional expansion force was proposed ten years ago to explain flat rotation curves without requiring dark matter. This force has been applied to various astronomical or cosmological situations. More recently, it has been shown that it could result from Bohmian Quantum mechanics, bridging the gap between it and cosmology and leading to a universe evolution different from that of the Standard Model. In particular, the expansion rate was found to depend on the observation distance, which could shed new light on the Hubble controversy. In this article, this point is studied in the context of a very simple model of a homogeneous universe that would be either finite or infinite, and with a view to providing a precise test of the H measurements. It is also shown that the universal expansion force has decreased over time, as assumed by the "early dark energy" hypothesis. Will the effect of this force allow us to resolve the controversy? In any case, the results of this new approach can allow us to test the validity of the theory itself as well as the finitude or infinity of the universe.

Keywords: Bohmian Quantum Mechanics; expansion; Hubble Constant; infinite Universe ; finite Universe ; early dark energy.

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1 Introduction

Despite the capabilities of the Standard Model, our current understanding of the Universe faces two major challenges: the lack of proof of the existence of Dark Matter and the fact that the value of the Hubble Constant depends on the experimental measurement methods [1 to 5].

The expansion force theory [6] defines an additional force to Newtonian attraction, without modifying it, unlike MOND theory [7].

Considered as the expression of a new principle of dynamics, this force has been applied to the study of various astrophysical situations, including the study of an expanding inhomogeneous universe [8 to 10], the hypothesis of a mass erosion phenomenon [11], the application to the planar structures of dwarf galaxies [12], studies leading to the idea that gravity would be dual, with a well-known attractive part and a repulsive part (the expansion force at the origin of the expansion of the universe, which would require the presence of negative matter [8, 10]).

Moreover, it has been shown that this force would result from a quantum phenomenon, within the framework of Bohmian Quantum Mechanics, during the interaction of two elementary particles associated with their pilot waves [13]. The radial part of this force do generate the expansion of the universe and its lateral part explains the observed excess rotational speeds [14].

This quantum approach, applied to the dynamics of a distant galaxy (and its associated pilot wave) interacting with all the masses of the universe, has made it possible to construct a model of the universe evolution that differs from the Standard Model: a universe assumed to be flat, homogeneous, and infinite, which, moreover, has neither beginning nor end. In this framework, the expansion rate has been shown to depend on the observation distance or (equivalently) on the time corresponding to this distance.

We do examine here in more detail the variation laws of the expansion rate, with a view to provide an element of response to the Hubble controversy, depending on whether the universe—assumed to be very simple: flat and homogeneous—appears to be "finite" or "infinite".

After a review of the generation of this model, we will examine more closely these two cases.

2 Review of the expansion force as a Bohmian Quantum effect

To model the attraction of a distant galaxy by all the masses of the universe using Bohmian Quantum Mechanics [13], a pilot wave (assumed to be Gaussian) was associated with this galaxy, considered as a "particle" subject to the potential generated by all the masses of the universe. For a homogeneous universe of density ρ_0, we thus obtained the following dynamical equation, for the position r of this "particle":

$$\frac{dr}{dt} = \frac{rt}{t^2+\tau^2} - \frac{MG}{r^2}t \tag{1}$$

$$\text{with } M(r) = \frac{4}{3}\pi\rho_0 r^3 \tag{2}$$

and the time parameter τ given by:

$$\tau = \frac{2m\text{Д}^2}{\hbar} \tag{3}$$

Д being the width of the pilot wave (supposed to have a gaussian shape).

The time parameter τ being extremely large, eq (1) was easily approximated as:

$$\frac{dr}{dt} = -Krt \tag{4}$$

where
$$K = \frac{4}{3}\pi G\rho_0 \tag{5}$$

We note here that it is the Newtonian term which prevails in eq. (1).

Furthermore, it was verified that this approach is equivalent to the introduction of an expansion force complementary to Newton's attraction, since the following relationship can be deduced from equations (2), (4), and (5):

$$\ddot{r} = -\frac{MG}{r^2} + \frac{\dot{r}^2}{r} \tag{6}$$

The solution to eq. (4) then led to the following universe dynamics:

$$r = Re^{-K\frac{t^2}{2}} \tag{7}$$

Independently of considerations related to the evolution of the universe - detailed in [14] - this approach notably concludes with the following evolution of the expansion rate:

$$H = \frac{\dot{r}}{r} = -Kt \tag{8}$$

[Present time t_0 has been denoted negatively, because we live in an epoch of expansion, later followed by an epoch of contraction of the universe]

3 "Finite" or "infinite" universe?

The important point to consider here is that it is the Newtonian term that governs the dynamics of the universe: the particle is attracted by all masses of the universe, and everything happens as if these (internal) masses were gathered at the center of the sphere in question (at the observer's level).

However, it should be noted that Gauss's theorem is only valid outside this sphere.

If, conversely, we were to consider a (simplistic) universe that was perfectly homogeneous and infinite, any particle located anywhere in this universe would obviously see all Newtonian contributions cancel out in opposite pairs. It is therefore clear that the above analysis does not apply to a homogeneous infinite universe, but rather to a homogeneous finite universe.

In a homogeneous infinite universe, there can therefore be no Newtonian term, and the following dynamical equation must apply:

$$\frac{dr}{dt} = \frac{rt}{t^2+\tau^2} \tag{9}$$

In reality, the degree of cancellation of the Newton's force will depend on many aspects (local or global, degree of homogeneity, curvature, etc.). This is why quotation marks are used to evoke "finite" or "infinite" universes.

In both cases, our center of interest is the expansion rate variation.

4 Case of a "finite" universe

In this case, the results of the previous study [13] apply, with the universe dynamics given by equation (7) and the linear variation of H:

$$H = \frac{\dot{r}}{r} = -Kt \tag{8}$$

which are illustrated in Figures 1 and 2.

The expansion is also characterized by the acceleration coefficient:

$$q = \frac{r\ddot{r}}{\dot{r}^2} \tag{10}$$

which is estimated to have the value ½ at the present time t_0. This implies:

$$H_0 = -\frac{2}{t_0} = \sqrt{2K} \tag{11}$$

Differentiation as a function of t around t_0 (or of r around r_0 via eq. (7)) highlights which variations of the Hubbel Constant should currently be observed at first degree:

$$\frac{dH}{H_0} = \frac{dt}{t_0} = \frac{dr}{2r_0 L\left(r_0/R\right)} \tag{12}$$

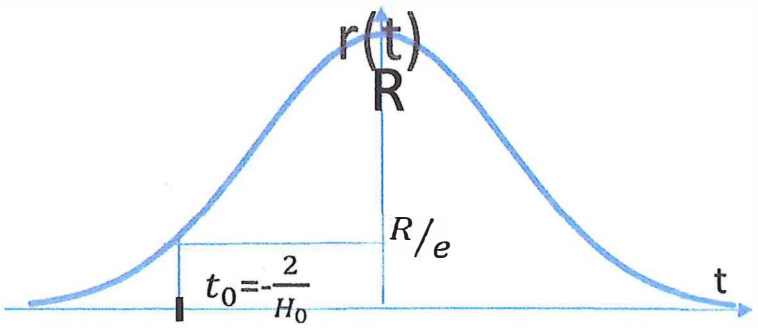

Fig. 1: Dynamics of a "finite" universe (Eq. (7))

We see that this model differs from the standard model: there is no Big Bang, and the expansion phase will be followed by a contraction phase.

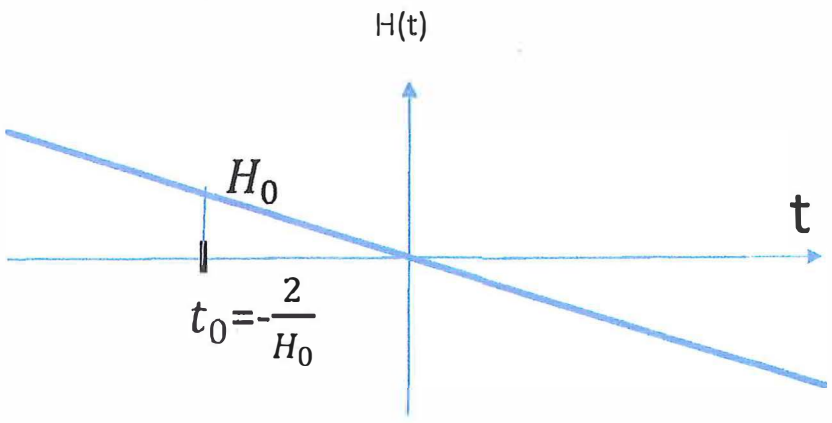

Fig. 2: Evolution of the expansion rate for a "finite" universe (Eq (8))

Recall that the present time is negative: we are on the expansive part of the universe evolution; more precisely, before the inflection point of the curve r(t) on the left, which represents the currently accelerated expansion. We see that H is a decreasing function of r or t. In other words, this model predicts that the measurement of H should decrease with the observation distance, as is indeed the case with the present measurement results, which are weaker for the CMB [4,5].

5 Case of an "infinite" universe

In this case, equation (9) leads to the following dynamics:

$$r = c\sqrt{t^2 + \tau^2} \tag{13}$$

where c is the speed of light (note that the speed $\dot{r}$ is always less than c and tends towards c for an infinite time).

The expansion rate then varies according to:

$$H = \frac{t}{t^2+\tau^2} = \frac{c}{r^2}\sqrt{r^2 - c^2\tau^2} \tag{14}$$

And the calculations of differential increases around (r_0, t_0) lead to:

$$\frac{dH}{dt} = H_0^2\left(\frac{\tau^2}{{t_0}^2} - 1\right) = \frac{c^2}{{r_0}^2} - 2H_0^2 \tag{15}$$

$$\frac{dH}{dr} = \frac{1}{r_0}\left(\frac{1}{t_0} - 2H_0\right) = \frac{c}{r_0^3}\frac{\left(2c^2\tau^2 - {r_0}^2\right)}{\sqrt{{r_0}^2 - c^2\tau^2}} \tag{16}$$

The variations of r and H are illustrated in figures 3 and 4.

The dynamics of an infinite universe therefore includes a contraction phase followed by an expansion phase. It also does not include a Big Bang (at most a minimal shrinkage at time 0). In this sense, our proposals fall under the theories of universes without beginning or end [15 to 17].

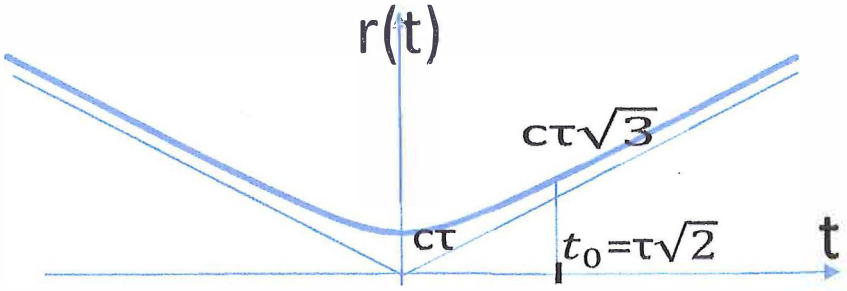

Fig. 3: Dynamics of an "infinite" universe (Eq. (13))

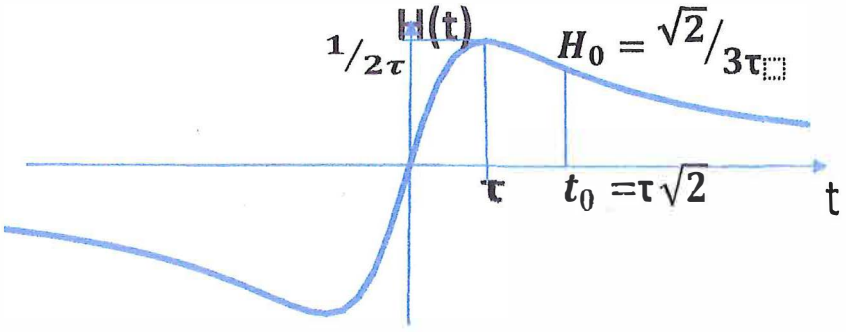

Fig. 4: Evolution of the expansion rate for an "infinite" universe (Eq (14))

The expansive part of curve (13) means that we are at a positive time t_0 and the current acceleration coefficient, of value ½, implies here the relationship between t_0 and τ:

$$t_0 = \tau\sqrt{2} \tag{17}$$

with $$H_0 = \frac{\sqrt{2}}{3\tau} = \frac{2}{3t_0} \tag{18}$$

We are therefore located after the maximum of H and we see that H is still a decreasing function of r and t.

6 Universal expansion force

It should be noted that, unlike equation (7) specifying the expansion force for a "finite" universe, the calculation of $\ddot{r}$ for an "infinite" universe leads to:

$$\ddot{r} = \frac{\tau^2}{t^2}\frac{\dot{r}^2}{r} \tag{19}$$

In this case, the expansion force to be considered undergoes an additional temporal variation.

This result seems to call into question our initial intuition [6] of an expansion force equal to $\frac{\dot{r}^2}{r}$, which explains flat rotation curves.

In reality, the difference between our two idealizations lies in the fact that the second ("infinite" universe) assumes conditions such that Newton's force can be neglected by large-scale symmetry: this obviously depends on the considered entire environment and, in particular, on the conditions of homogeneity, curvature, and topology.

It is clear that at the level of the motion of stars in a galaxy, Newton's force must be considered, justifying our expansion force in $\frac{\dot{r}^2}{r}$. On the other hand, at the level of the expansion of the universe that interests us here, we can imagine that the conditions of homogeneity, curvature, and topology agree so that our simplifying assumptions called "infinite universe" do apply, therefore leading to eq. (19), which can be called the "universal expansion force":

$$\ddot{r} = \frac{\tau^2}{t^2}\frac{\dot{r}^2}{r} = \frac{c\tau^2}{(t^2+\tau^2)^{3/2}} \qquad (19)$$

7 Expansion force versus early dark energy

Let's examine the variation over time of this universal expansion force.

We see that it decreases over the expansive period, being in fact inversely proportional to the volume of the universe r^3. For $t_0 = \tau\sqrt{2}$, it would today be about 5 times weaker than for t=0.

This result seems to be in good agreement with the recent hypothesis called the "early dark energy field" proposed by Riess and Kamionkowski [18 to 20] according to which "the baby universe was expanding faster than we think" and "an additional repulsive force that was pushing against space at the time and has since died out."

According to my theory, this force has not totally died out. It has been caused by a Bohmian Quantum effect, its energy would come from a continuous erosion process, and it should be considered as a dual aspect of gravity, whose effect gradually diminishes with the expansion of the universe.

After ten years of rejection, these ideas perhaps deserve attention.

8 Conclusion

The present theory does indeed predict an evolution of the expansion rate as a function of the observation distance (or corresponding time). Formulas (8) and (12) theoretically govern the predicted evolutions in both cases of a "finite" or "infinite" universe, as do formulas (14) and (15, 16) for the differential versions. These developments are therefore likely to provide concrete tests for the estimates of H in the context of the Hubble controversy.

We also observe that the theory of the "infinite" universe establishes a relationship between H and τ, that is, a bridge between long-distance cosmology and quantum mechanics. Since τ depends on the spatial extensions of galaxies (Д), it is conceivable that these extensions evolve with expansion and time. The

precise variation law will obviously result from a more complete modeling of a more real universe (with curvature and inhomogeneities).

If there were to be a concordance between the experimental data from H measurements and the propositions in this article, this would have the merit of providing an attempt to resolve the Hubble controversy, as well as an element of validation of the expansion force theory. Furthermore, we have shown that the expansion force varies with time and that, for an "infinite" universe, it would be about 5 times smaller than at the beginning of the expansion. This result suggests that it could be linked to the recent idea of the "early dark energy field", or even support it more generally [18 to 20]

As for the issue of a finite or infinite universe, the debate obviously remains open. As mentioned above, the terms "finite" or "infinite" (written in quotation marks) are based on whether or not Newton's force is taken into account, as it can be "cancelled" by symmetry. Incidentally, intermediate situations can be envisioned, in which the effect of this force would be more or less reduced, depending on the actual conditions of the universe's unfolding and its real characteristics: local or global homogeneity defects, partial curvatures, etc. The corresponding models could provide as many more refined derivations of the contribution to the expansion force.

As here formulated, the present propositions presuppose an absence of curvature and the underlying assumption of a very simple internal topology, which tends to conflate the concepts of finiteness and the fact that the universe has no boundary. In particular, since measurements of the CMB [4,5] have revealed a non-zero, but weak, curvature, our approach can be considered from this point of view as a first approximation.

Much modeling work will still be necessary to bring our simple idealization closer to a more real universe, and perhaps definitively resolve the Hubble controversy as well as the question of Dark Matter.

Isn't the real Universe much richer than our approximations claim [21]?

References

[1] Freedman, W.L. & al., The Astrophys. J., 882:34 (29pp), (2019 Sept 1)

[2] Freedman, W.L. & al., The Astrophys. J., 891:57(14pp), (2020 March 1).

(3) A. G. Riess *et al.*, *A comprehensive measurement of the local value of the Hubble constant with 1 km s^{-1} Mpc^{-1} uncertainty from the Hubble space telescope and the SH0ES team*, Astrophys. J. Letters, (2022).

(4) Joao Alves J., Forveille T., Pentericci L, Shore S., *Planck 2018 results*, VI Cosmological Parameters, Astronomy & Astrophysics, Volume 641, Sept.2020
[5] Di Valentino E., Melchiorri A., *Planck evidence for a closed Universe and a possible crisis for cosmology*, arXiv:1911.02087v1 [astro-ph.CO],(5 Nov 2019).

[6] Fleuret, J., *Towards a new generalized space expansion dynamics applied to the rotation of galaxies and Tully Fisher law*, Astrophys. Space Sci., 350-2 (2014).

[7] Milgrom M., The Astrophys. J., 270:365-370, (1983 July 15)

[8] Fleuret, J., *Cosmic expansion acceleration and negative matter*, Physics Essays, 32, 2, (2019) pp. 263-269.

[9] Fleuret, J., *Gravity and dual gravity: proposals for an inhomogeneous expanding universe* (2020),

https://bf592c95-d85f-4684-ab5a-52a5d6c8840c.filesusr.com/ugd/ce6eb5_c2d48be3c2284d2c833304b3c0786631.pdf

[10] Fleuret, J., *Expansion acceleration versus Dark Matter: additional comments (2021).*

https://bf592c95-d85f-4684-ab5a-52a5d6c8840c.filesusr.com/ugd/ce6eb5_0491824fa116438693b516811bb035b5.pdf

[11] Fleuret, J., *Expansion as a consequence of a rest-mass erosion theory,* Astrophys Space Sci (2015) 357:68, DOI 10.1007/s10509-015-2253-3

[12] Fleuret J., *New Expansion Dynamics Applied to the Planar Structures of Satellite Galaxies and Space Structuration*, Journal of Modern Physics, Vol.07 No.16(2016), Article ID:72940,9 pages, 10.4236/jmp.2016.716204

[13] Fleuret J., *Expansion force as a Bohmian Quantum effect on elementary particles*, Hadronic Journal 46, 135 - 142 (2023)

[14] Fleuret J., *Expansion force as a consequence of Bohmian Quantum Mechanics*, Hadronic Journal 44, 343 – 355 (2021)

[15] Lynds P., *On a finite universe with no beginning or end* (2006), https://arxiv.org/pdf/physics/0612053

[16] Lynds P., *Why there is something rather than nothing: The finite, infinite and eternal,* (2012), https://arxiv.org/pdf/1205.2720

[17] Damien A. Easson D.A., Lesnefsky J.E., *Eternel Universes*, Physical Review D 112, 063545 (2025)

[18] Poulin V., Smith T.L.; Karwal T., *Early Dark Energy can Resolve the Hubble Tension*, Physical Review Letters, V.122, Issue 22 (June 2019)

[19] Kamionowski M., Riess A.G., *The Hubble Tension and Early Dark Energy*, Annual Review of Nuclear and Particle Science, Volume 73, Issue , pp. 153-180 (sept. 2023)

[20] Riess, Kamionkowski, *A Weird Form of Dark Energy might solve a cosmic conundrum*, Scientific American, (oct. 15, 2024), https://www.scientificamerican.com/article/could-early-dark-energy-resolve-the-mystery-of-cosmic-expansion/

[21] Luminet *J.P. Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background*, arXiv:astro-ph/0310253 (2003)

HADRONIC JOURNAL 48, 451 - 471 (2025)
DOI 10.29083/HJ.48.04.2025/SC451

COHOMOLOGICAL APPROACH TO HAMILTONIAN SYSTEMS

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Abstract

BRST formulation of cohomological Hamiltonian mechanics is presented. In the path integral approach, we use the BRST gauge fixing procedure for the partition function with trivial underlying Lagrangian to fix symplectic diffeomorphism invariance. Resulting Lagrangian is BRST and anti-BRST exact and the Liouvillian of classical mechanics is reproduced in the ghost-free sector. The theory can be thought of as a topological phase of Hamiltonian mechanics and is considered as one-dimensional cohomological field theory with the target space a symplectic manifold. Twisted (anti-)BRST symmetry is related to global $N = 2$ supersymmetry, which is identified with an exterior algebra. Landau-Ginzburg formulation of the associated $d = 1$, $N = 2$ model is presented and Slavnov identity is analyzed. We study deformations and perturbations of the theory.

1 INTRODUCTION

Recently, path integral approach to classical mechanics has been developed by Gozzi, Reuter and Thacker [1]-[10]. The BRST approach to the model has been suggested [34], with the result that it is an example of cohomological quantum field theory [11]-[15], which can be derived by an appropriate BRST gauge fixing of a theory in which the underlying Lagrangian is zero. An extensive literature exists on the topological field theories, which have been investigated within the BRST quantization scheme; see ref. [14] for a review.

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The most close examples of cohomological field theory to the one considered in this paper are Witten's topological non-linear sigma model [12], in which a basic field is the map from Riemannian surface to a fixed Kahler manifold as the target space, and topological Landau-Ginzburg models [18]. Also, one-dimensional sigma model with the target space a compact Kahler manifold having nontrivial homotopy group π_1 has been considered recently by Cecotti and Vafa [19].

Problem of giving the BRST formulation itself of the cohomological classical mechanics is drastically simpler than that of the forementioned topological field theories, which are characterized by rich field content and a set of symmetries, since we need to fix only one symmetry. Also, the theory is one-dimensional that makes general structure of the theory less complicated than that of the two- or higher dimensional topological field theories; for example, there are no conventional "instanton corrections" and the notion of spin is irrelevant. Furthermore, it should be stressed that the resulting theory is essentially a *classical* one. As the result, sharing many properties with the usual topological quantum field theories, it differs from those by absence of quantum ($\hbar$) corrections. However, we should to emphasize that there arisés a major distinction from the conventional topological field theories studied in the literature since symplectic structure of the target space rules out quadratic terms from Lagrangian, leading to a first-order character of the system. Such specific theories, describing Hamiltonian systems, are worth to be studied exclusively.

In view of the above, cohomological classical mechanics represents perhaps the simplest example of topological field theory. So, after constructing the BRST invariant Lagrangian the focus of our paper is to study the implications and novel aspects arising from the BRST approach and associated superymmetry. The work in this paper enables us to use (supersymmetric) field theory as a way of deeper understanding of Hamiltonian systems. In general, this approach provides a powerful tool to investigate the fundamental properties and characteristics of Hamiltonian systems such as ergodicity, Gibbs distribution, Kubo-Martin-Schwinger condition, integrability, and Lyapunov exponents. Particularly, we think it is illuminating and instructive to map out some identifications one can draw between the topological field theories and Hamiltonian systems. This is, in part, to establish dictionary between the old and modern techniques used in studying classical dynamical systems.

Another motivation of our study is that we consider the BRST formulation of cohomological classical mechanics as providing a basis to give BRST

formulation of (cohomological) quantum mechanics and, from then on, apply topological field theory methods to study quantum mechanical systems. The key to making the connection between them lies in treating quantum mechanics as a smooth $\hbar$-deformation of the Hamiltonian one, within the phase space (Weyl-Wigner-Moyal) formulation of quantum mechanics [41]. In addition, there arises a tempting possibility to give a classification of possible topologies of constant energy submanifolds of the phase space [40] for the case of reduced Hamiltonian systems. Of course, this idea is reminiscent of the one of using BRST symmetry and supersymmetry to obtain various topological results.

As to examples of field theoretic approach to Hamiltonian systems we notice that the path integral approach to Euler dynamics of ideal incompressible fluid viewed as Hamiltonian system has been developed recently by Migdal [20], to study turbulence phenomenon in terms of the path integral over the phase space configurations of the vortex cells. Also, Niemi and Palo [21] considered classical dynamical systems using $d = 2$, $N = 2$ supersymmetric nonlinear sigma models. They followed studies on the Arnold conjecture on the number of T-periodic trajectories [22] by Floer [23].

In this paper, we present some results of elaborated analysis of the model made in ref. [24], outline of which is as follows. We start with a target space interpretation of Hamiltonian mechanics and explore the BRST gauge fixing scheme to fix diffeomorphism invariance of the trivial underlying Lagrangian. The gauge fixing condition is Hamilton's equation. When both the BRST and anti-BRST symmetries are incorporated the Lagrangian exhibits Z_2 symmetry. Liouvillian of ordinary classical mechanics is reproduced by the associated Hamilton function, in the ghost-free sector. The model reveals symplectic structure represented by using of the cotangent superbundle over phase space naturally supplied by the field content. Then we analyse Slavnov identity to demonstrate that the model is perturbatively trivial and BRST anomaly free. With this set up, we study in some detail the associated supersymmetric model and cohomology. Namely, we use topological twist of the BRST and anti-BRST operators to obtain global $N = 2$ supersymmetry, and relate the supersymmetry to an exterior algebra. In so doing, we are able to identify physical states of the theory. The link between the BRST and anti-BRST symmetries is the supersymmetric (Ramond) ground states of the associated $d = 1$, $N = 2$ model which are found to be in correspondence with cohomology classes of the target manifold. The states can be in general treated as (cohomology classes of) form valued classical probability distributions on the phase space M^{2n}. In the

ghost-free sector, they correspond to the probability distribution related to conventional ergodic Hamiltonian systems. Physically relevant (normalizable) solutions of the supersymmetry equations are given specifically by the Gibbs state form coming from the $2n$-ghost sector. We stress that the supersymmetry appears to be a strong constraint on the physics. Criterion for regular/nonregular motion regimes in Hamiltonian systems is related to the Witten index known as a measure for supersymmetry breaking. Partition function evaluates Euler characteristic of the target space and the Witten index is equal to Euler characteristic, too (not surpising result, certainly, obtained earlier in the context of topological $d = 2$, $N = 2$ sigma models). Also, we find that Poincare integral invariants can be naturally identified as homotopically trivial BRST invariant observables. Existence of homotopically nontrivial Poincare invariants is a consequence of the field theoretic approach. We discuss briefly on the connection of the model to Morse theory observing that Hamiltonian may serve as a Morse function and then proceed to obtain Landau-Ginzburg formulation of the $d = 1$, $N = 2$ model using a superspace technique. One of the results is that the model admits Landau-Ginzburg desription [12, 18] so that its properties can be largely understood in terms of superpotential. The lowest component of the superpotential has been identified as Hamiltonian. The action appears to be in the form of a D-term. The ring of chiral operators consists of polynomials modulo the relation characterizing critical points of the Hamiltonian flow. We show that the time reparametrization invariance of the model requires the fundamental homotopy group $\pi_1(M^{2n})$ to be nontrivial and conjecture that it can be broken by the topological effect of changing cohomological classes of a symplectic two-form. Valuable information comes from studying possible deformations and perturbations of the action. We analyze deformations of the superpotential and symplectic tensor. What is most interesting is that the supersymmetry preservation conditions for the deformation of superpotential (Hamiltonian) by analytic function is explicitly related to integrability properties of the system. This is another step toward revealing connection between the supersymmetry and integrals of motion. We also treat the deformation of the coordinate dependent fundamental symplectic tensor and found that to preserve the supersymmetry it must be a variation of the symplectic structure. We study BRST invariant observables and its correlation functions. The BRST invariant observables of interest are closed p-forms on symplectic manifold and correspond to the de Rham cohomology classes. Elaborating connection between the BRST symmetry and supersymmetry, we identify the BRST invariant observables with chiral

operators of the $d = 1$, $N = 2$ model. The anti-BRST observables can be treated in the same manner. Also, there arise naturally homotopy classes of classical periodic orbits so that coefficients of the p-forms take values in the linear bundles of appropriate representations of $\pi_1(M^{2n})$. This leads to consideration of the loop space consisting of mappings $S^1 \to M^{2n}$ which is a natural object in the topological framework. We remark that periodic orbits are in many ways the key to the classical dynamics. The correlation functions are found to be related to the intersection number, with the two-point correlation function, in the homotopically trivial sector, representing the standard intersection form in cohomology identified as the topological metric. As it is in the topolological field theories, there are no "local" degrees of freedom in the theory under consideration that means that the correlation functions are not time dependent. A certain kind of correlation functions interwines the BRST and anti-BRST sectors and are known to be related to the Lyapunov exponents, positive values of which are strong indication of chaos in Hamiltonian system. The p-form sectors, Z_p, $p = 0, \ldots, 2n$, of the partition function for the periodic orbits are evaluated via realizing M^{2n} as a covering space and using monodromy.

2 THE BRST FORMULATION

A Hamiltonian dynamical system can be described geometrically by a phase space manifold M^{2n} equipped by a symplectic form ω and a Hamiltonian H [27]. The evolution of the system in time t is given by a particular set of trajectories on M^{2n}, parametrized by t, such that Hamilton's equation holds. On the other hand, Hamiltonian dynamical system can be described as one-dimensional field theory, in which dynamical variable is the map, $a^i(t)$: $M^1 \to M^{2n}$, from one-dimensional space M^1, $t \in M^1$, to $2n$-dimensional symplectic manifold M^{2n}. The commuting fields $a^i = (p_1, \ldots, p_n, x^1, \ldots, x^n)$ are local coordinates on the target space, the phase space M^{2n}, endowed with a non-degenerate closed two-form ω; $\omega^n \neq 0$, $d\omega = 0$. In terms of the local coordinates, $\omega = \frac{1}{2}\omega_{ij} da^i \wedge da^j$; $\omega_{ij}\omega^{jk} = \delta_i^k$. We take ω_{ij} to be constant symplectic matrix, *i.e.* use canonical (Darboux) coordinates.

Starting point is the partition function $Z = \int D[a] \exp iI_0$, where $I_0 = \int dt \mathcal{L}_0$ and the Lagrangian is trivial, $\mathcal{L}_0 = 0$. The symmetry of the action we are interested in is symplectic diffeomorphism invariance, which leaves the symplectic tensor ω_{ij} form invariant,

$$\delta a^i = \ell_h a^i. \tag{2.1}$$

Here, $\ell_h = h^i\partial_i$ is a Lie-derivative along the Hamiltonian vector field, $h^i = \omega^{ij}\partial_j H(a)$ [27].

By introducing the ghost field $c^i(t)$ and the anti-ghost field $\bar{c}_i(t)$, we write the BRST version of the diffeomorphism (2.1),

$$sa^i = c^i, \quad sc^i = 0, \quad s\bar{c}_i = iq_i, \quad sq_i = 0, \tag{2.2}$$

where the BRST operator s is nilpotent, $s^2 = 0$, and q_i is a Lagrange multiplier. By an obvious mirror symmetry to the BRST transformations (2.2), we demand the anti-BRST transformations hold.

The BRST and anti-BRST invariant Lagrangian is $\mathcal{L} = \mathcal{L}_0 + s\bar{s}B$, with B being of ghost number zero. With the choice $B = \omega_{ik}a^i(\partial_t a^k - h^k)$, we find the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_0 + q_i\partial_t a^i - a^i\partial_t q_i + i(\bar{c}_i\partial_t c^i + c_i\partial_t\bar{c}^i) - q_i h^i + i\bar{c}_i\partial_k h^i c^k. \tag{2.3}$$

Lagrangian (2.3) is invariant under the following Z_2 symmetry: $t \to -t$, $a^i \to a^i$, $q_i \to -q_i$, $c^i \to c^i$, $\bar{c}_i \to -\bar{c}_i$, $h^i \to -h^i$.

Hamilton function, $\mathcal{H}$, associated with the Lagrangian (2.3) can be readily derived,

$$\mathcal{H} = q_i h^i + i\bar{c}_i c^k\partial_k h^i. \tag{2.4}$$

In the ghost-free part, it covers the usual Liouvillian, $L = -h^i\partial_i$, of ordinary classical mechanics derived in the operational formulation of classical mechanics by Koopman and von Neumann [28]. The Hamilton function $\mathcal{H}$ is a generalization of the Liouvillian to describe evolution of the form valued probability distribution, $i\partial_t\rho(a,c) = \mathcal{H}\rho(a,c)$, instead of the usual distribution function (zero-form) governed by the Liouville equation, $\partial_t\rho(a) = -L\rho(a)$.

Recall that the phase space M^{2n} is usually considered as a cotangent bundle over configuration space M^n, $x^\alpha \in M^n$. Analogously, let us consider cotangent bundle M^{4n} over M^{2n} endowed by symplectic two-form $\tilde{\Omega}$ with local cooordinates $\tilde{y}^c = (q_i, a^j)$; $c,d = 1,\ldots,4n$. M^{4n} can be thus viewed as the second generation phase space with a base space M^{2n} and natural projection $p : M^{4n} \to M^{2n}$ provided by $(q_i, a^j) \mapsto (0, a^j)$. Enlarging the bundle M^{4n} by a Grassmannian part with coordinates $(c^i, \bar{c}_j)$, we define the cotangent superbundle $M^{4n|4n}$ equipped by block diagonal supersymplectic matrix $||\Omega_{ab}|| = \mathrm{diag}(||\tilde{\Omega}_{cd}||, ||E_{cd}||)$, where E is $4n \times 4n$ unit matrix, with local coordinates $y^a = (\tilde{y}^d, c^i, \bar{c}_j)$; $a,b = 1,\ldots,8n$. The graded Poisson brackets in $M^{4n|4n}$ can be defined by using Ω as $\{F,G\}_\Omega = (\partial_a F)\Omega^{ab}(\partial_b G)$,

where F and G are functions on $M^{4n|4n}$ and $\partial_a = \partial/\partial y^a$; $\Omega_{aa'}\Omega^{a'b} = \delta_a^b$. We note that this is, in fact, a nontrivial result because of emerging of the supersymplectic structure Ω having no counterpart in the Cartan calculus. The probability distribution forms ρ on M^{2n} can be viewed as functions on $M^{4n|4n}$, with the graded Poisson bracket algebra being the algebra of classical observables. The Hamilton function (2.4) is ad_H operator acting on functions on $M^{4n|4n}$ and represents horizontal vector field in the fiber bundle. The Schrodinger-like equation for evolution of the distribution form can be rewritten in a Hamiltonian form, $i\partial_t\rho = \{\mathcal{H},\rho\}_\Omega$.

The analysis [24] of the Slavnov identity showed that the model is perturbatively trivial and BRST anomaly free.

3 THE $d = 1$, $N = 2$ MODEL

The possible physical states, $\rho \equiv |\mathrm{phys}\rangle$, are found as solutions of the system of equations consisting of the BRST and anti-BRST cohomology equations [7, 32, 33], $Q\rho = 0$, $\bar{Q}\rho = 0$. They are equivalence classes of appropriate Q and $\bar{Q}$ cohomologies, $\rho \sim \rho + Q\rho' + \bar{Q}\rho''$. To study the physical states, we exploit the identification of the twisted (anti-)BRST operator algebra with $N = 2$ supersymmetry. Using the twisted BRST and anti-BRST operators

$$Q_\beta = e^{\beta H}Qe^{-\beta H} = Q - \beta c^i\partial_i H, \quad \bar{Q}_\beta = e^{-\beta H}\bar{Q}e^{\beta H} = \bar{Q} + \beta\bar{c}_i\omega^{ij}\partial_j H, \tag{3.1}$$

where $\beta \geq 0$ is a real parameter, one can easily find that they are conserved nilpotent supercharges and their anticommutator closes on the Hamilton function, $\{Q_\beta, \bar{Q}_\beta\} = 2i\beta\mathcal{H}$. Consequently, these supercharges, together with the Hamilton function $\mathcal{H}$, build up a global $N = 2$ supersymmetry. The supersymmetry transformations leaving the Hamilton function $\mathcal{H}$ invariant are

$$\delta_s a^i = c^i, \quad \delta_s c^i = 0, \quad \delta_s\bar{c}_i = iq_i - \beta\partial_i H, \quad \delta_s q_i = -i\beta c^k\partial_i\partial_k H, \tag{3.2}$$

$$\bar{\delta}_s a^i = \omega^{ki}\bar{c}_k, \quad \bar{\delta}_s c^i = i\omega^{ik}q_k + \beta\omega^{ik}\partial_k H, \quad \bar{\delta}_s\bar{c}_i = 0, \quad \bar{\delta}_s q_i = i\beta\omega^{jk}\bar{c}_j\partial_i\partial_k H. \tag{3.3}$$

In general, the supersymmetry garantees that there will be a set of exactly degenerate ground states. More specifically, if the supersymmetry is exact the Hamiltonian system is in the non-regular motion regime since, in this case, there is only one conserved entity (energy) while if the system is in regular motion regime, *i.e.* there is at least one additional nontrivial integral of motion, the supersymmetry is always broken. However, there is a

subtlety to make one-to-one correspondence. Namely, broken supersymmetry does not necessary imply regular motion regime and, also, non-regular motion regime does not rule out broken supersymmetry. It seems that the problem lies precisely in possible degeneracy of the ground state, that is, $\ker \mathcal{H}$ modulo cohomological equivalence may consist of several elements. The space $\ker \mathcal{H}$ is finite dimensional since $\mathcal{H}$ is identified with an elliptic operator. The indication of this is that an ergodic Hamiltonian system is characterized just by *non-degenerate* zero eigenvalue solution of the Liouville equation. So, one is led to study physics coming from the degenerate vacuum. In the following, we assume there is a discrete set of supersymmetric ground states that corresponds to the case of an elliptic operator on compact manifold.

The relation of the BRST and anti-BRST symmetries of the original theory to the $N = 2$ supersymmetry is due to supersymmetric (Ramond) ground state sector of the associated $d = 1$, $N = 2$ supersymmetric model, in accordance with the topological-antitopological fusion by Cecotti and Vafa [31].

3.1 COHOMOLOGY

Our next step in studying the physical states is relating of the $N = 2$ supersymmetry to an elliptic complex:

$$
\begin{gathered}
d_\beta \leftrightarrow Q_\beta, \quad \bar{d}_\beta \leftrightarrow \bar{Q}_\beta, \\
\Delta_\beta \equiv d_\beta \bar{d}_\beta + \bar{d}_\beta d_\beta \leftrightarrow \{Q_\beta, \bar{Q}_\beta\} = 2i\beta\mathcal{H}, \\
(-1)^p \leftrightarrow (-1)^C,
\end{gathered}
\tag{3.4}
$$

where the exterior (co-)derivative $d_\beta = d + \beta c^i \partial_i H$ ($\bar{d}_\beta = d^* - \beta \bar{c}_i \omega^{ij} \partial_j H$) acts on p-forms, $\rho \in \Lambda^p$, and C is the ghost number. The d_β cohomology groups, $H^p(M^{2n}) = \{\ker d_\beta / \mathrm{im}\, d_\beta \cap \Lambda^p\}$, are finite when M^{2n} is compact. According to Hodge theorem, canonical representatives of the cohomology classes $H^p(M^{2n})$ are harmonic p-forms, $\Delta_\beta \rho = 0$. They are closed p-forms, $d_\beta \rho = 0$, $\bar{d}_\beta \rho = 0$. One can then define B_p as the number of independent harmonic forms, $B_p(\beta) = \dim\{\ker \Delta_\beta \cap \Lambda^p\}$. Formally, B_p continuosly varies with β but, being a discrete function, it is, in fact, independent on β so that one can find B_p by studying the vacua of the Hamilton function (2.4), $\mathcal{H}\rho = 0$. Due to the supersymmetry, it can be rewritten as $Q_\beta \rho = 0$, $\bar{Q}_\beta \rho = 0$, and defines the Ramond sector. If we consider deformations δa^i along the solution of Hamilton's equation, then in order for $a^i + \delta a^i$ to still

be a solution it has to satisfy the deformation equation $\partial_t \delta a^i = \delta h^i$. This equation is the equation for the conventional Jacobi fields, first variations, which are tangent to the target manifold, $\delta a^i \in TM^{2n}$, and can be thought of as the "bosonic zero modes". As it is common in topological field theories, these modes are just compensated by anti-commuting zero modes through the ghost dependent term, in the Lagrangian (2.3). Indeed, the associated equation of motion for ghosts, $\partial_t c^i = \partial_k h^i c^k$, represents BSRT variation of Hamilton's equation. Topologically, the relation of the supersymmetry equations to the cohomology equations can be readily understood by taking into account the fact that the twist (3.1) is a homotopy operation.

According to the identification (3.4) with the exterior algebra, the physical states are harmonic p-forms on the target space M^{2n}. In the standard de Rham complex, the B_p's are simply Betti numbers, with the alternating sum, $\chi = \sum^{2n}(-1)^p B_p$, being Euler characteristic of the symplectic manifold M^{2n}. Due to the identifications (3.4), it is then straightforward to show that the Witten index [30], $\mathrm{Tr}(-1)^C$, is just the Euler characteristic of M^{2n} (cf. [10]). So, we conclude that the criterion for the regular/nonregular regimes in Hamiltonian systems is the Witten index. To break supersymmetry the Witten index needs to be zero.

Recent studies [7, 32, 33, 34] of the physical states showed that physically relevant (normalizable) solutions to the generalized ergodicity equation come from the $2n$-ghost sector and have specifically the form of the Gibbs state, $\rho = \kappa K^n \exp[-\beta H]$ where $K = \frac{1}{2}\omega_{ij} c^i c^j$ and κ is a constant. It is important to note that under the field redefinition this state transforms as $2n$-form rather than as a scalar. The reason of this lies in the cohomology. The other ghost sectors yield solutions of the form $\rho = \kappa K^p \exp[+\beta H]$, for the even-ghost sectors, $p = 2, 4, \ldots, 2n-2$, and those either trivial or not depending on β, for the odd-ghost sectors. It is assumed that $\rho(a, c)$ must be normalizable in each p-ghost sector, *i.e.* $\int \rho(a, da) da^1 \wedge \cdots \wedge da^p = 1$, $p = 0, \ldots, 2n$.

In the limit $\beta \to 0$, we recover the classical Poincare integral invariants [27] corresponding to K^p, $p = 1, \ldots n$, as solutions of the supersymmetry equations, which are indeed invariants under Hamiltonian flow, $\mathcal{H} K^p = 0$. In particular, the $2n$-ghost K^n, which we are viewing as cohomological representative of the unit, corresponds to the volume form ω^n of the phase space conservation of which is statement of the Liuoville theorem. They are fundamental BRST invariant (topological) observables of the theory, $\{Q, K^p\} = 0$, and form the classical cohomology ring, $K^{n+1} = 0$. Similarly, the conjugates of the Poincare invariants, $\bar{K}^p$, are the anti-BRST invariant

(anti-topological) observables, $\{\bar{Q}, \bar{K}^p\} = 0$, $p = 1, \ldots n$, which are powers of the Poisson bivector $\bar{K}$.

3.2 CONNECTION TO MORSE THEORY

There appears to be no relation of the $d = 1$, $N = 2$ model to Morse theory [37] similar to that found in Witten's supersymmetric quantum mechanics [29], because of the absence of the term quadratic in h^i, in the Hamilton function (2.4), which would play a role of the potential energy. An immediate reason is that the symplectic two-form ω is *closed* so that this does not allow us to construct, or obtain by the BRST procedure, non-vanishing terms in the Hamilton function quadratic in fields, except for the ghost-antighost term. Although the Hamilton function (2.4) is linear in the commuting fields, it contains the Hessian of the Hamiltonian $H(a)$, which can serve as Morse function, the number of isolated critical points of which are known to be related to Euler characteristic. The critical points here are simply stationary points of the Hamiltonian flow, $h^i = 0$, with the number of critical points being $\sum_{dH=0} sign(\det ||\partial_i \partial_j H(a)||)$.

3.3 THE LANDAU-GINZBURG FORMULATION

Collection of fields composing a Landau-Ginzburg type system is the following: real field a^i, two anticommuting real fields c^i and $\bar{c}_i$, real field q^i, and superpotential W solely responsible for the "interaction" terms. We choose a single superfield as follows [25]:

$$X^i(t, \theta_1, \theta_2) = a^i(t) - i\theta_1 c^i(t) + i\theta_2 \omega^{ij} \bar{c}_j(t) + i\theta_1 \theta_2 \omega^{ij} q_j(t), \tag{3.5}$$

where θ_I, $I = 1, 2$, are real anticommuting parameters, and the component fields are nothing but a collection of the BRST doublets. A manifestly covariant Lagrangian

$$\mathcal{L} = \int d\theta_1 d\theta_2 \left\{ \frac{1}{2} \omega_{ij} X^i D_1 D_2 X^j + iW(X) \right\}, \tag{3.6}$$

can be written with the help of the covariant derivatives in the superspace with local coordinates (t, θ_1, θ_2),

$$D_1 = \frac{\partial}{\partial \theta_1} + i\theta_2 \frac{\partial}{\partial t}, \quad D_2 = \frac{\partial}{\partial \theta_2} + i\theta_1 \frac{\partial}{\partial t}, \tag{3.7}$$

$D_1^2 = D_2^2 = 0$, $\{D_1, D_2\} = 2i\partial_t$, and the superpotential $W(X)$, which is a real analytic function of X. In terms of the component fields, we find the Lagrangian (3.6) in the form

$$\mathcal{L} = q_i\partial_t a^i - a^i\partial_t q_i + i(\bar{c}_i\partial_t c^i + c^i\partial_t\bar{c}_i) + \frac{\partial W(a)}{\partial a^i}\omega^{ij}q_j - i\omega^{ij}\bar{c}_i\frac{\partial^2 W(a)}{\partial a^k\partial a^j}c^k, \tag{3.8}$$

where $W(a)$ is the lowest component of the superpotential. With the identification $W(a) = H(a)$, the Lagrangian (3.8) covers the original Lagrangian (2.3). So, we conclude that one can start with the one-dimensional Landau-Ginzburg $N = 2$ model (3.6) and obtain via topological twist the cohomological theory (2.3) with already gauge fixing. Note that the Lagrangian (3.6) provides the action to be of the form of a D-term.

For completeness, let us to note that the whole set of BRST and anti-BRST transformations takes the form of the following constraint: $(s_I - \partial/\partial\theta_I)X^i = 0$, where we have denoted $s_1 = s$, $s_2 = \bar{s}$, saying that the BRST and anti-BRST operators can be treated as derivatives in the odd coordinates of superspace. Then, the topological invariance of the action is obvious in superspace because of supertranslation invariance, $X^i(t, \theta_1, \theta_2) \to X^i(t, \theta_1 + \theta_1', \theta_2 + \theta_2')$, of the Berezin integration.

It is simple but important consequence of supersymmetry algebra that the action with the Lagrangian (3.6), like any D-term since it is the highest component, can be written *both* as $I = \{Q_\beta, \xi\}$ and $I = \{\bar{Q}_\beta, \bar{\xi}\}$, where ξ is some fermionic (odd) field integrated over time. Note also that we were not forced to use a F-term, which is defined as an integral over only half of superspace, to reproduce the original action.

The above are the essential ingredients necessary for arguments in analyzing implications of the supersymmetric structure of cohomological Hamiltonian mechanics. It is wellknown that the symplectic two-form ω can not be in general defined to be constant globally on M^{2n} so the question arises as to cohomology classes of ω. Two choices of the Lagrangian, for which ω are in different cohomology classes, differ by F-term. Since the action is of the form of a D-term we have no topological effect of changing the classes which could be in principle considered by perturbing the action by a F-term. The problem on cohomological class $[\omega]$ of ω in the theory has a direct link to time reparametrization invariance. Consider the time reparametrization $t \to e^{\phi}t$, where ϕ is a parameter ($d = 1$ Lorentz transformation). In the Lagrangian (3.6), the only effect it produces is the scaling $\omega \to e^{-\phi}\omega$. This implies scaling of the volume of compact phase space, $V = \int_{M^{2n}}\omega^n \to e^{-n\phi}V$, so that

with the factor $1/V^{1/n}$ the Lagrangian (3.6) becomes time reparametrization invariant. Without loss of generality, assume that ω is an exact two-form in some region $U \subset M^{2n}$, that is, $\omega = d\vartheta$, where $\vartheta = \vartheta_i da^i$ is a symplectic one-form. The general consistency requirement is then that the Wilson loop integral $\exp[i \oint_{\partial D} \vartheta]$ should not depend on a disk D, for which ∂D is its boundary. Hence, we must have $\int_{S^2} \omega = 2\pi k$, where k is an integer number. The scaling of ω demands $k = 0$, so we arrive at the conclusion that unless the condition

$$\int_{S^2} \omega = 0, \qquad \forall S^2 \subset M^{2n}, \tag{3.9}$$

is satisfied, the time reparametrization invariance of the theory is broken. The condition (3.9) is known in the literature [21, 23] and implies that the fundamental homotopy group $\pi_1(M^{2n})$ must be nontrivial. In other words, necessary condition for unbroken time reparametrization invariance of the model is that there should be non-contractible loops in M^{2n} for which, particularly, the Wilson loop integrals build up a representation. Inclusion of the topological effect of changing the cohomology classes of ω could lead to broken time reparametrization invariance.

The most basic elements of the given $d = 1$, $N = 2$ theory are analogues of the chiral and anti-chiral rings of the (topologically twisted) two-dimensional $N = 2$ models [31]. Since we are originally interested in the topologically twisted $N = 2$ model only the ground states are kept, so it is simple to make identification of these with the operators. Namely, the chiral operators, ϕ_i, are defined as the ones satisfying $[Q_\beta, \phi_i] = 0$, and the anti-chiral operators, $\bar{\phi}_i$, satisfy $[\bar{Q}_\beta, \bar{\phi}_i] = 0$. They are irreducible representations of the supersymmetry algebra. Then, we can make a correspondence between the Ramond ground states and chiral fields by choosing a canonical ground state $|0\rangle$, with the identification $\phi_i|0\rangle = |i\rangle + Q_\beta|\lambda\rangle$. Similarly, there is a natural isomorphism between the anti-chiral fields and the adjoint states $|\bar{i}\rangle$. In terms of the Landau-Ginzburg formulation, the chiral ring consists of the polynomials modulo the relation $dW = 0$, which defines critical points of the Hamiltonian vector field. The inner product on the space of the ground states corresponding to the fields ϕ_i and $\phi_{\bar{j}}$ is $g_{i\bar{j}} = \langle \bar{j}|i\rangle$, and geometrically plays the role of a metric in the associated Hilbert space while $\langle i|j\rangle$ gives rise to the topological metric, $\eta_{ij} = \langle \phi_i \phi_j \rangle_{top}$. The real structure matrix, $M = g\eta^{-1}$, relates the ground states with its adjoints, $\langle \bar{k}| = \langle j|M^j_{\bar{k}}$.

3.4 DEFORMATIONS AND PERTURBATIONS

Let us consider deformation of the Hamiltonian, $H(a) \to H(a) - \delta t_P P(a)$, where $P(a)$ is a local polynomial, and δt_P is a coupling constant parametrizing the deformation. This leads immediately to the following perturbation of the action: $\int dt\mathcal{L} \to \int dt\mathcal{L} + \delta t_P \int dt\mathcal{O}_P$, where

$$\mathcal{O}_P = iq_i\omega^{ij}\partial_j P(a) - \bar{c}_i\omega^{ij}\partial_k\partial_j P(a)c^k. \tag{3.10}$$

Direct calculations show that $\mathcal{O}_P$ is BRST and anti-BRST closed, $s\mathcal{O}_P = \bar{s}\mathcal{O}_P = 0$; see Eq.(3.11) below, with $\beta = 0$.

Non-trivial topological perturbation may have place only if the deformation term is not BRST exact cocycle. It can be readily checked that, in our case, $\mathcal{O}_P = s\{\bar{c}_i\omega^{ij}\partial_j P(a)\}$, so that there are no *nontrivial* topological perturbations coming from the deformation of the superpotential.

However, supersymmetry appears to be sensitive to the deformation. Using the definitions (3.1) and (3.10) we find directly

$$[Q_\beta, \mathcal{O}_P] = -i\beta c^k\partial_k(\omega^{ij}\partial_i H\partial_j P), \quad [\bar{Q}_\beta, \mathcal{O}_P] = i\beta\omega^{mn}\bar{c}_m\partial_n(\omega^{ij}\partial_i H\partial_j P). \tag{3.11}$$

Sufficient condition to both the commutators in (3.11) vanish is that the Poisson bracket

$$(\partial_i H)\omega^{ij}(\partial_j P) \equiv \{H, P\}_\omega = \text{const}, \tag{3.12}$$

or, more precisely, is equal to a locally constant function. For the case of connected M^{2n}, this equation is necessary and sufficient condition for the Hamiltonian flows defined by the functions P and H to commute, with P viewed as another Hamiltonian.For the phase space with nontrivial π_1 one should use here the notion of local Hamiltonian vector field. We note that, in general, when one knows a Hamiltonian flow commuting with the flow under study it is possible to construct an integral of motion [27]. Hence, the equations (3.11) represent a link between the supersymmetry and integrability.

When examining the formal evolution of P we see that P linearly changes with time, $dP/dt = \text{const}$. For compact connected M^{2n} the polynomial $P(a)$ is bounded so that the constant must be zero, *i.e.* P is an integral of motion, $\{H, P\}_\omega = 0$. It has been argued [7] that the existence of an additional *nontrivial* integral of motion leads to *broken* supersymmetry. In view of this, polynomial P should be a trivial integral of motion to preserve the supersymmetry. For noncompact M^{2n} the polynomial P is not necessarily bounded so that we are left with the general condition (3.12). The same is true for

M^{2n} with nontrivial π_1 and also when P is an analytic function. However, in general, if (polynomial or analytic function) P is in involution with H it must be a *trivial* first integral. From the above analysis we conclude that the symmetries do not fix the Lagrangian uniquely, with nontrivial supersymmetry preserving perturbation term (3.10), where P satisfies (3.12) with *non-zero* constant, can be added to the action. However, in the case of compact connected M^{2n} there are no nontrivial supersymmetry preserving perturbations supplied by the deformation with polynomial.

Let us turn to considering of deformation of the symplectic tensor. Under an infinitesimal change $\omega_{ij} \to \omega_{ij} + \epsilon_{ij}$, one sees from (3.1) that Q_β is invariant whereas $\bar{Q}_\beta$ changes by

$$\delta\bar{Q}_\beta = [Q_\beta, \bar{K}_\epsilon] = \bar{c}_i\epsilon^{ij}(iq_j + \beta\partial_j H), \tag{3.13}$$

where $\bar{K}_\epsilon = \frac{1}{2}\epsilon^{ij}\bar{c}_i\bar{c}_j$; $\epsilon_{ij}\epsilon^{jk} = \delta_i^k$. Using (3.13) one finds that the Hamilton function changes by

$$\delta\mathcal{H} = \frac{1}{2i\beta}\{Q_\beta, [Q_\beta, \bar{K}_\epsilon]\} = q_i\epsilon^{ij}\partial_j H + i\bar{c}_i\partial_k(\epsilon^{ik}\partial_j H)c^j. \tag{3.14}$$

In order for the $N = 2$ supersymmetry algebra to be preserved, $\bar{K}_\epsilon$ must commute with $\bar{Q}_\beta$,

$$[\bar{Q}_\beta, \bar{K}_\epsilon] = \omega^{kl}[q_l, \epsilon^{ij}]\bar{c}_i\bar{c}_j\bar{c}_k. \tag{3.15}$$

This commutator vanishes if and only if ϵ_{ij} are coefficients of closed two-form so that it appears to be the case of a variation of the symplectic structure. As a consequence, this variation preserves also the BRST and anti-BRST symmetries.

When one attempts infinitesimal change by *non-constant* tensor, $\epsilon_{ij} = \epsilon_{ij}(a)$, the previous arguments break down because the second equality in (3.13) does not hold, and, even more, the anti-BRST operator receives non-nilpotent contribution. This case, however, is important since, as it was mentioned above, constant symplectic tensor can not be in general globally defined on a symplectic manifold. For instance, on a compact one, for which case one uses a covering by local chartes with constant ω_{ij} owing to Darboux theorem. Analysis made on this generalized model [6, 32, 33] has shown that with a slight modification of the anti-BRST operator the ISp(2) algebra is regained, with all the basic results of the constant symplectic tensor case being reproduced. The BRST approach to this generalized model can be readily developed in the same fashion as it for the case of Darboux coordinates. So, with the infinitesimal change, $\omega_{ij}(a) \to \omega_{ij}(a) + \epsilon_{ij}(a)$, the

first equalities in Eqs. (3.13) and (3.14), where $\mathcal{H}$ and $\bar{Q}_\beta$ are generalized accordingly, are still valid while Eq.(3.15) becomes

$$[\bar{Q}_\beta, \bar{K}_\epsilon] = \{\omega^{kl}(a)[q_l, \epsilon^{ij}(a)] + \frac{1}{2}\epsilon^{li}(a)\partial_l\omega^{jk}(a)\}\,\bar{c}_i\bar{c}_j\bar{c}_k, \tag{3.16}$$

with the final result is that, again, $\epsilon(a)$ must be a closed two-form to preserve the $N = 2$ supersymmetry.

4 BRST OBSERVABLES AND CORRELATION FUNCTIONS

In general, the BRST invariant observables of interest are of the form

$$\mathcal{O}_A = A_{i_1\cdots i_p}(a)c^{i_1}\cdots c^{i_p}, \tag{4.1}$$

which are p-forms on M^{2n}, $A \in \Lambda^p$. One can easily find that $\{Q, \mathcal{O}_A\} = 0$ if and only if A is closed since $\{Q, \mathcal{O}_A\} = \mathcal{O}_{dA}$. Consequently, the BRST observables correspond to the de Rham cohomology and form a classical cohomology ring $\mathcal{R}$ of M^{2n} which corresponds to the ring of chiral operators ϕ_i. The BRST observables are related immediately to the BRST invariant states, via construction analogous to that of relating the chiral fields with the supersymmetric ground states. The distribution forms are identified with differential forms as

$$A_{i_1\cdots i_p}(a)c^{i_1}\cdots c^{i_p}|0\rangle \leftrightarrow A_{i_1\cdots i_p}(a)da^{i_1}\wedge\cdots\wedge da^{i_p}. \tag{4.2}$$

In terms of the vacuum distribution forms, the isomorphism between the chiral fields and states in the Ramond sector becomes more explicit. Namely, the Hilbert space of the model consists of all square summable p-forms, $|A_i\rangle = |i\rangle$, with coefficients taking value in some linear bundle E on which the operators ϕ_i corresponding to the cohomology classes act by wedge product. We note that since the flow equation is real the complex conjugate of the vacuum distribution form A_i should be again a vacuum distribution form, and thus can be expressed as a linear combination of the vacuum distribution forms.

The basic field $a^i(t)$ is characterized by homotopy classes of the map $M^1 \to M^{2n}$. Clearly, these are in general classes of the map $S^1 \to M^{2n}$, that is, the classes of conjugated elements of the fundamental homotopy group[1]

[1] S^1 and M^{2n} with marked points give rise to relative homotopy group π_1, which can be used here as well.

$\pi_1(M^{2n})$ while $M^1 = R^1$ is trivial. These classes can be weighted with different phases and controlled by some coupling parameter, in a standard way. The fields are then characterized by appropriate representations σ of the group $\pi_1(M^{2n})$, which we assume to be nontrivial. Therefore, the coefficients of the cohomology ring $\mathcal{R}$ take values in the linear bundles E_σ associated to the representations σ.

Thus, we should study specifically periodic orbits in M^{2n} characterized by a period T, $a^i(0) = a^i(T)$. In general, it is suitable to consider the fields defined on circle, with circumference T. For that case we write $t \in S^1$. In fact, the T-periodic solutions to Hamilton's equation are elements of the loop space of Hamiltonian system which is an object of recent studies [21, 22, 23] on infinite dimensional version of Morse theory. When necessary one can replace circle by the real line by taking the limit $T \to \infty$. We remark that periodic orbits are presumably dense in phase space and at finite time scale may mimic typical dynamics arbitrary well. Moreover, the families of periodic orbits have the unique property that they continue smoothly across the fractal boundary between the regular region and the chaotic region, with stable and unstable character in these regions respectively, being thus the only unifying agents between these two disparate regions.

Let us now turn to consideration of the correlation functions of the BRST invariant observables (4.1). If N is a closed submanifold of (compact) M^{2n} representing some homology class of codimension m ($2n$-m cycle), then, by Poincare duality, we have m-dimensional cohomology class A (m-cocycle), which can be taken to have delta function support on N [38]. Thus, any closed form A is cohomologious to a linear combination of the Poincare duals of appropriate N's. The general correlation function is then of the form,

$$\langle \mathcal{O}_{A_1}(t_1) \cdots \mathcal{O}_{A_m}(t_m) \rangle, \tag{4.3}$$

where A_k are the Poincare duals of the N's. Our aim is to find the contribution to this correlation function on S^1 coming from a given homotopy class of the map $S^1 \to M^{2n}$. The conventional techniques with the moduli space $\mathcal{M}$ [11] consisting of the fields $a^i(t)$ of the above topological type can be used here owing to the BRST symmetry. Namely, the non-vanishing contribution to (4.3) can only come from the intersection of the submanifolds $L_k \in \mathcal{M}$ consisting of a's such that $a^i(t) \in N_k$, and we obtain familiar formula [39],

$$\langle \mathcal{O}_{A_1}(t_1) \cdots \mathcal{O}_{A_m}(t_m) \rangle_{S^1} = \#\Big(\sum^m \cap L_k\Big), \tag{4.4}$$

relating the correlation function to the number of intersections. As it was expected, the correlation functions do not depend on time but only on the indeces of the BRST observables.

Homotopically trivial sector. This sector is characterized by a's which are homotopically constant maps, $[a^i(t)] = [a^i(t_0)]$, and, therefore, the correlation function (4.4) can be presented as $\int A_1 \wedge \cdots \wedge A_m$. The standard Poincare integral invariants, K^p, identified as fundamental BRST observables correspond to the homotopically trivial sector since they have been originally formulated in the Darboux coordinates, in which ω_{ij} are constant coefficients. Also, two-point correlation function can be used to define the topological metric, $\eta_{ij} \equiv \langle A_i | A_j \rangle = \int A_i \wedge A_j$, which is just the intersection form in the cohomology (cf. ref. [19]). Particularly, it is easy to check that in canonical basis of the forms, the action of the real structure matrix M reads, $*A_j^* = g_{i\bar{j}} A_i$, and the topological metric is $\eta_{ij} = \delta_{ij}$.

Existence of homotopically nontrivial Poincare invariants follows from the fact that globally $\omega_{ij} = \omega_{ij}(a)$ and there are the classes of a^i. Nontrivial character of these invariants is due to the fact that cohomology class of ω on compact M^{2n} is necessarily nontrivial.

The BRST observable (4.1) can be naturally understood as $(p,0)$-form corresponding to the general (p,q)-form,

$$\mathcal{U}_A = A^{j_1 \cdots j_q}_{i_1 \cdots i_p}(a) c^{i_1} \cdots c^{i_p} \bar{c}_{j_1} \cdots \bar{c}_{j_q} |0\rangle \leftrightarrow A^{j_1 \cdots j_q}_{i_1 \cdots i_p}(a) da^{i_1} \wedge \cdots \wedge da^{i_p} \partial_{j_1} \wedge \cdots \wedge \partial_{j_q}, \tag{4.5}$$

which is a function on $M^{4n|4n}$, where the anti-ghosts represent the anti-BRST sector. Accordingly, we associate the $(0,q)$-forms to the anti-BRST observables, which can be considered in the same manner as the BRST ones. A particular kind of the observable (4.5) interwining the BRST and anti-BRST sectors has been studied recently by Gozzi and Reuter [8], $\mathcal{U}_A = \delta(a(t_0) - a_0) c^{i_1}(t) \cdots c^{i_p}(t) \bar{c}_{i_1}(t_0) \cdots \bar{c}_{i_p}(t_0) |0\rangle$. After normal ordering, this observable can be thought of as the operator creating p ghosts from Fock vacuum (p-volume form in TM^{2n}) at some time t_0 and point $a_0 \in M^{2n}$, and then annihilating them at some later time t. The correlation function

$$\langle \mathcal{U}_A \rangle_{S^1} = \langle \bar{A}(t) A(t_0) \rangle_{top} \equiv \Gamma_p(T, a_0), \tag{4.6}$$

does not depend on specific time, and Γ_p depends only on the period T. A profound result [8] is that the higher order ($p \geq 1$) largest Lyapunov exponents can be extracted from this correlation function.

The p-form sector of the partition function with appropriate periodic boundary conditions and the fields defined on circle, $Z_p(T) = \mathrm{Tr}_{S^1} \exp[-i\mathcal{H}_p t]$,

can be expressed in terms of Γ_p in the following normalized form, $Z_p(T) = \mathrm{Tr}_{S^1}\Gamma_p(t,a)/\mathrm{Tr}_{S^1}1$, where Tr_{S^1} denotes the path integral over all the a's, which are T-periodic solutions to Hamilton's equation. The problem in computing $Z_p(T)$ arises due to the fact that the integral over all the a's does not converge. Finite value can be evaluated when realizing that the integral receives contributions from the homotopy classes of the a's mentioned above so that we need to subtract excessive degrees of freedom. Below, we perform explicit computation by realizing M^{2n} as a covering space, $f : M^{2n} \to Y^{2n}$, of a suitable linearly connected manifold Y^{2n} having the same fundamental group as it of S^1, $\pi_1(Y^{2n},t_0) \simeq \pi_1(S^1,t_0)$. Since $\pi_0(Y^{2n},t_0) = 0$ the set of preimages is *discrete*, $f^{-1}(t_0) = \{a_0, a_1, \ldots\}$. The number of elements of $f^{-1}(t_0)$ and of the monodromy group given by the factorization, $G = \pi_1(Y^{2n},t_0)/f_*\pi_1(M^{2n},a_0)$, coincides due to canonical one-to-one correspondence between the set $f^{-1}(t_0)$ and the monodromy group G, and does not depend on t_0 due to $\pi_0(Y^{2n},t_0) = 0$. Hence, $Z_p(T) = \sum\limits_{g\in G}\Gamma_p(T,a_g)$, which is finite if G is a finite group. Clearly, the Z_p's are topological entities, which can be used to define *topological* entropy [8] of Hamiltonian systems. Also, in terms of Morse theory it has been shown [4, 10] that the partition function $Z(T)$ at $T \to 0$ localizes to critical points of H so that the sum $\sum(-1)^p Z_p$ is the number of T-periodic solutions to Hamilton's equation and is equal to Euler characteristic of M^{2n}. Moreover, recent studies [21] showed that if Hamiltonian H is a *perfect* Morse function on M^{2n}, for which case one has to have $H^{2k+1}(M^{2n}) = 0$, it saturates the lower bound in the Arnold conjecture, which states that the number of non-degenerate contractible T-periodic solutions of Hamilton's equation is greater or equal to sum of Betti numbers of M^{2n}.

References

[1] E.Gozzi and M.Reuter, Phys. Lett. **B 233**, 383 (1989).

[2] E.Gozzi, M.Reuter and W.D.Thacker, Phys. Rev. **D 40**, 3363 (1989).

[3] E.Gozzi and M.Reuter, Phys. Lett. **B 238**, 451 (1990).

[4] E.Gozzi and M.Reuter, Phys. Lett. **B 240**, 137 (1990).

[5] E.Gozzi, Progr. Theor. Phys. Suppl. **111**, 115 (1993).

[6] E.Gozzi, M.Reuter, and W.D.Thacker, Phys. Rev. **D 46**, 757 (1992).

[7] E.Gozzi, M.Reuter and W.D.Thacker, Chaos, Solitons and Fractals **2**, 441 (1992).

[8] E.Gozzi and M.Reuter, *"Lyapunov exponents, path integrals and forms"*, INFN preprint UTS-DFT-92-32 (1992); Chaos, Solitons and Fractals 4 (1994) (to appear).

[9] E.Gozzi and M.Reuter, *"Metaplectic spinor fields on phase space: a path-integral approach"*, INFN preprint INFN/AE-93/10 (1993).

[10] E.Gozzi, *"Counting periodical trajectories via topological classical mechanics"*, INFN preprint INFN/AE-93/32 (1993).

[11] E.Witten, Comm. Math. Phys. **117**, 353 (1988).

[12] E.Witten, Comm. Math. Phys. **118**, 411 (1988); IASSNS preprint IASSNS-HEP-93/3.

[13] E.Witten, *"Introduction to cohomological field theories"*, IASSNS preprint IASSNS-HEP-90/57 (1990).

[14] D.Birmingham, M.Blau and M.Rakowski, Phys. Rep. **209**, 129 (1991).

[15] B.A.Dubrovin, Comm. Math. Phys. **141**, 195 (1992).

[16] E.Witten, Nucl. Phys. **B 340**, 281 (1990).

[17] M.J.Perry and E.Teo, *"BRS and anti-BRS symmetry in topological Yang-Mills theory"*, DAMTP preprint DAMTP R92/33 (1992).

[18] C.Vafa, Mod. Phys. Lett. **A 6**, 337 (1991).

[19] S.Cecotti and C.Vafa, *"Ising model and $N = 2$ supersymmetric theories"*, Harvard Univ. preprint HUTP-92/A044 (1992).

[20] A.A.Migdal, *"Loop equation in turbulence"*, Princeton Univ. preprint PUTP-1383 (1993); *"Turbulence as statistics of vortex cells"*, Princeton Univ. preprint PUTP-1409 (1993).

[21] A.J.Niemi and K.Palo, *"On the characterization of classical dynamical systems using supersymmetric nonlinear σ-models"*, CERN preprint CERN-TH.7512/94 (1994), hep-th/9412023; see also K.Palo, Phys. Lett. **B 321**, 61 (1994); A.Yu.Morozov, A.J.Niemi, and K.Palo, Nucl. Phys. **B 377**, 295 (1992).

[22] H.Hofer and E.Zehnder, *Symplectic invariants and Hamiltonian dymanics* (Birkhauser Verlag, 1994).

[23] A.Floer, Duke Math. J. **53**, 1 (1986); Comm. Math. Phys. **118**, 215 (1988); J. Diff. Geom. **18**, 513 (1988); Comm. Pure Appl. Math. **41**, 775 (1988); **42**, 775 (1989).

[24] A.K.Aringazin, V.V.Arkhipov, and A.S.Kudusov, Hadronic J. **18** (1995) (to appear).

[25] A.K.Aringazin, V.V.Arkhipov, and A.S.Kudusov, Hadronic J. **18** (1995) (to appear).

[26] L.Baulieu and I.M.Singer, Nucl. Phys. Proc. Sup. **15 B**, 12 (1988).

[27] R.Abraham and J.Marsden, *Foundations of mechanics* (Benjamin, New York, 1978); V.I.Arnold, *Mathematical methods of classical mechanics* (Springer, 1978).

[28] B.O.Koopman, Proc. Nat. Acad. Sci. USA **17**, 315 (1931); J.von Neumann, Ann. Math. **33**, 587; 789 (1932).

[29] E.Witten, J. Diff. Geom. **17**, 661 (1982).

[30] E.Witten, Nucl. Phys. **B 202**, 253 (1982).

[31] S.Cecotti and C.Vafa, Nucl. Phys. **B 367**, 359 (1991); *"Topological anti-topological fusion"*, Harvard Univ. preprint HUTP-91/A031 (1991).

[32] A.K.Aringazin, Hadronic J. **16**, 385 (1993).

[33] A.K.Aringazin, Phys. Lett. **B 314**, 333 (1993).

[34] A.K.Aringazin, K.M.Aringazin and V.V.Arkhipov, Hadronic J. **17**, 429 (1994).

[35] E.Witten, *"Mirror manifolds and topological field theory"*, in: *Essays on mirror manifolds*, ed. S.-T.Yau (International Press, 1992).

[36] V.I.Arnold and A.Avez, *Ergodic problems of classical mechanics* (Benjamin, New York, 1968).

[37] J.Milnor, *Morse theory*, Ann. of Math. Studies **51** (Princeton Univ. Press, 1973).

[38] R.Bott and L.Tu, *Differential forms in algebraic topology* (Springer, Berlin, 1982).

[39] E.Witten, Int. J. Mod. Phys. **A 6**, 2775 (1991).

[40] A.T.Fomenko, *"Topology of constant energy surfaces of integrable Hamiltonian systems, and obstructions to integrability"*, Izv. AN SSSR, Matematica **50**, 1276 (1986); A.T.Fomenko and H.Tsishang, *"On typical topological properties of integrable Hamiltonian systems"*. Izv. AN SSSR, Matematica **52**, 300 (1988); see also R.Bott, *"On topological obstructions to integrability"*, Actes Congr. Int. Mat. **1**, 27 (1970).

[41] E.Gozzi and M.Reuter, *"Quantum-deformed geometry on phase-space"*, DESY preprint DESY 92-191 (1992); *"A proposal for a differential calculus in quantum mechanics"*, Trieste Univ. preprint UTS-DFT-93-10 (1993); *"Quantum deformed canonical transformations, W_∞-algebras and unitary transformations"*, Trieste Univ. preprint UTS-DFT-93-11 (1993). A.K.Aringazin, K.M.Aringazin, S.Baskoutas, G.Brodimas, A.Jannussis and K.Vlachos, *"q-Deformed harmonic oscillator in phase space"*, Karaganda Univ. preprint KSU-DTP 93/10 (1993). A.K.Aringazin, *"BRST formulation of cohomological quantum mechanics in phase space"*, in preparation.

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THE DISCRETE SYMMETRIES AND THE LIST OF HAMILTONIANS OPERATORS OF INTEGRABLE SYSTEMS

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Abstract

General properties of discrete transformations of integrable systems are discussed. It proposed the new group-theoretical interpretation for discrete transformation which allow to describe all essential properties of integrable systems as a direct corollary of theory of group representations of discrete transformations. The list of the pairs of Hamiltonians operators is presented for the most often used $(1+1)$ integrable systems.

1 Introduction

Roughly speaking, the most characteristic properties of $(1+1)$ completely integrable systems can be described as follows. Every such system has infinite number of involutive conservation laws, which can be written as

$$\frac{\partial I_0^n}{\partial t} + \frac{\partial I_1^n}{\partial x} = 0. \tag{1.1}$$

Each time component I_0^n can be considered as a Hamiltonian density, generating with the help of the corresponding Poisson brackets a completely integrable equation. The totality of these equations form a so-called hierarchy of integrable equations. To find the explicit form of a hierarchy it is necessary to know all conserved quantities I_0^n. There are two main methods to find conserved quantities. One of them based on the inverse scattering

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method, consists of concerning a recurrence relation allowing to get consequently all conserved quantities, starting from a few initial ones. The other method uses the fact that any completely integrable equation, can be written using different Poisson brackets and different Hamiltonians. The different Poisson brackets are constructed with the help of the corresponding Hamiltonian operators. Knowing two Hamiltonian operators, forming a Hamiltonian pair [1], we can find all conserved quantities, starting again from a few initial ones.

These facts were known for a long time. However, there were no simple constructive methods for finding integrals and Hamiltonian pairs. The problem of finding of Hamiltonian operators changed radically after discovering the fact that all the most important properties of completely integrable systems, including their explicit solutions, are the direct consequence of their discrete symmetry. It became clear that the requirement of the invariance of a Hamiltonian operator with respect to the discrete symmetry is a powerful method of constructing the corresponding hierarchy of Hamiltonian structures.

In papers [3] the invariance condition was used to formulate a constructive procedure to find coefficients of the expansion of a Hamiltonian operator over powers of the total space derivative operator D. The starting point of the calculations was there the fixation of the maximal positive degree of the operator D. The proposed method allowed, in principle, to find successfully all the required coefficients. It was essential to know the behaviour of the conserved quantities under the discrete transformation.

The goal of present paper as a consequence of analyze of the results of [3], to understand the general structure of Hamiltonian operators, their connection with conserved quantities and obtain explicit expressions for them in the case of $(1+1)$ integrable systems.

As the summary of our investigations,we would like to emphasize that the availability of Poisson structures is the distinctive property of $(1+1)$ integrable systems. The proper explanation for that fact is not discovered at present time.

2 Discrete transformation of integrable systems

All integrable systems under consideration are invariant with respect to discrete local invertible transformation describing by the substitution

$$\tilde{u} = \phi(u, u', u'', ..., u^{(r)}) \equiv \phi(u) \tag{2.1}$$

where u is s-dimensional vector function, u',u",.. its derivatives of the corresponding order with respect to "space" coordinates.

The property of invertibility means that equality (2.1) can be resolved and "old" function u may be written locally in terms of new functions $\tilde{u}$ and its derivatives.

Frechet derivative $\phi'(u)$ corresponding to substitution (2.1) is the $s \times s$ matrix operator defined as

$$\phi'(u) = \phi_u + \phi_{u'} D + \phi_{u''} D^2 + ... \tag{2.2}$$

where D^m is operator of m-time differentiation with respect to corresponding $u''^{..'}$ space coordinates. The reader can find more detail information about this object in [1].

Let us consider the equation

$$F_n(\phi(u) = \phi'(u) F_n(u) \tag{2.3}$$

which in some other different notations was considered firstly in [2]. where $F_n(u)$ is unknown s-component vector function, each component of which depends on u and its derivatives up to n-order.

The equation (2.3) possesses one obvious trivial solution $F_n(u) = u'$ for each substitution. In order to prove this it is sufficient to differentiate the equation (2.1) once with respect to some of its space coordinate.

If the equation (2.2) possesses some other solution (for a given $\phi(u)$) except of the trivial one, then we will call such substitution as the integrable substitution or mapping.

We emphasize once more that equation (2.2) contain two unknown functions $\phi(u)$ and $F_n(u)$ and only for narrow class of integrable substitutions it possesses nontrivial solutions for $F_n(u)$ function.

It is possible to connect the equation (system) of evolution type with each nontrivial solution of (2.2)

$$u_t = F_n(u) \tag{2.4}$$

which is obviously invariant with respect to substitution $u \to \phi(u)$. (It is easy to see that equation $u_t = u'$ is indeed invariant with respect to arbitrary substitution.)

Let us now compare the equation (2.3) with definition of linear representation $T(g)$ of some group (for definiteness Lee group)

$$\Phi(gx) = T(g)\Phi(x) \tag{2.5}$$

where g is the group element,$T(g)$ is the group operator for some representation, $\Phi(x)$ the basis of the corresponding representation space.

The obvious correspondence takes place after comparison (2.5) with (2.2)

$$\Phi(x) \to F_n(u), \quad T(g) \to \phi'(u)$$

If this correspondence has a deep group theoretical foundation then it is possible to anticipate that the different solutions of the equation (2.2) are connected by some linear transformation

$$F_n(u) = H^{n,n'} F_n(u) \tag{2.6}$$

Indeed all solutions (with different n) of the equation (2.2) (from this point of view) are the basis vectors of some group representation with the group operator $\phi'(u)$ and in the case of its irreducibility all possible basics of this representation are connected by linear transformation.

The exactly same situation takes place in the theory of $(1+1)$ integrable systems. Operators $H^{n,n'}$ in this theory are known as Hamiltonian operators of different degree. We will show below how to construct these operators using only discrete symmetry requirement.

3 General structure of Hamiltonian operators

The solutions of following two equations will be important for further consideration

$$\phi'(u)H(u)\phi'(u)^{-1} = H(\phi(u)), \quad \phi'(u)J(u)\phi'(u)^T = J(\phi(u)) \tag{3.1}$$

where $\phi'(u)^T = \phi_u^T - D\phi_{u'}^T + D^2\phi_{u''}^T - ...$, and $H(u), J(u)$ are unknown $s \times s$ matrix operators, the matrix elements of which are polynomial of some finite order with respect to operator of differentiation D (of its positive and negative degrees).

From (3.1) and (2.2) it follows immediately that if $F_n(u)$ is some solution of main equation (2.2) then $H^p(u)F_n(u)$ (p is arbitrary natural number) will be some other solution of the same equation.

The solution of the second equation (3.1) under additional condition of its skew symmetry may be connected (interpreted) as a Poisson structure which is invariant with respect to transformation of discrete symmetry. Skew symmetrical operators $J(u)$ are known as Hamiltonian ones. Two different solutions of the second equation (3.1) (if it is possible to find them), for instance $J_1(u)$ and $J_2(u)$ in combination $J_1 J_2^{-1}$ satisfy the first equation (3.1). Operator $J_1 J_2^{-1} J_1(u)$ is again the solution of the second equation (3.1) and so on. This is the way how in the theory of integrable systems usually Hamiltonian operators arise. It is necessary from independent assumption to find two different Poisson structures and after this fulfil describing above operation. In the problem of construction of Hamiltonian operators for integrable systems the equations (3.1) was firstly used in [3].

According to [3] it is possible to expect that (some partial) solutions of equations (3.1) may be represented in the form

$$J(u) = \sum_{\alpha,\beta} a_\alpha D^{-1} b_\beta + \sum_i A_i D^i$$

where a_α,b_β are some s-dimensional column (line) vectors, A_i are some $s \times s$ matrixes the components and matrixes elements of which are the functions of u and its derivatives.

In order to understand this structure let us consider the action of $\phi'(u)$ operator on solution of main equation (2.2) $F_n(u)$. We obtain

$$\phi'(u)F_n(u) = (\phi'(u)F_n(u)) + c_1 D + c_2 D^2 + ... = F_n(\phi(u)) + c_1 D + c_2 D^2 + ..$$

From this consideration it follows immediately that if the operator $J(u)$ chosen in the form

$$J(u) = \sum_{n,n'} F_n(u) D^{-1} F_{n'}(u) + \sum_i A_i(u) D^i$$

then for operator $\phi'(u)J(u)\phi'(u)^T$ we obtain

$$\phi'(u)J(u)\phi'(u)^T = \sum_{n,n'} F_n(\phi(u))D^{-1}F_{n'}(\phi(u)) + \sum_i \tilde{A}_i(u)D^i$$

and the second equation (3.1) is equivalent to equality $\tilde{A}_i(u) = A_i(\phi(u))$.

The above notice allows to obtain comparatively simple explicit expressions for operators J_0, J_1 for construction of which as a rule it is necessary to use except of trivial solutions u' of (2.2) some other simplest one u for example.

In the next section we shall represent the list of integrable substitutions with corresponding expressions for Frechet derivatives, explicit form of operators J_0, J_1 which as was explained before is sufficient to construct the whole hierarchy of equations which are all invariant with respect to transformation of given discrete symmetry. In other words it is possible to say that from the point of view of theory of integrable substitution this is the recurrent method for construction new solutions of equation 2.2) from some initial one.

4 The list of discrete substitutions and Hamiltonian operators corresponding to them

Here we give a list of integrable substitutions [4] together with the corresponding Hamiltonian operators J_0, J_1. It is possible to reconstruct the whole hierarchy of equations with the same discrete symmetry with the help of them. We represent also the explicit form of Frechet operator by help of which all results can be checked without any difficulties. All examples below are connected with two component vector function u where for first component we conserve notation u and the second one will be denoted by v. The simplest system of hierarchy will be represented in the beginning of the corresponding subsection.

It is very interesting that all known examples of integrable substitutions in the case of rational spectral parameter (in terminology of inverse scattering method) are connected with equations of Toda chains (infinite) or some its generalizations. Is it possible to obtain this result and how from condition of resolving of equation (2.2)?

4.1 Nonlinear Schrödinger substitutions.

4.1.1 Nonlinear Schrödinger hierarchy without the derivative

$$\dot{u} + u'' - 2u(uv) = 0 \quad -\dot{v} + v'' - 2v(uv) = 0$$

The direct and inverse substitution in this case have the form

$$\tilde{u} = \frac{1}{v}, \qquad \tilde{v} = v[vu - (\ln v)''] \tag{4.1}$$

$$v = \frac{1}{\tilde{u}}, \qquad u = \tilde{u}[\tilde{u}\tilde{v} - (\ln \tilde{u})'']$$

If it rewrite substitution (4.1) we will obtain the infinite chain of equations for unknown function $x_n = \ln v_n$

$$x_n'' = \exp(x_n - x_{n-1}) - \exp(x_{n+1} - x_n)$$

This is exact infinite Toda chain in its original form.

Frechet matrix of substitution is the following

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2(uv) - \frac{(v')^2}{v^2} + \frac{2v'}{v}D - D^2 \end{pmatrix}$$

Keeping in mind the results of the last section one can convinced by direct calculations that the following two operators J_0, J_1

$$J_0(u,v) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J_1(u,v) = \begin{pmatrix} 2uD^{-1}u & D - 2uD^{-1}v \\ D - 2vD^{-1}u & 2vD^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

4.1.2 Modified nonlinear Schrödinger hierarchy

$$\dot{u} + u'' + 2u'(uv) = 0 \quad -\dot{v} + v'' - 2v'(uv) = 0$$

The direct and inverse substitution in this case have the form

$$\tilde{u} = \frac{1}{v}, \qquad \tilde{v} = v[vu - (\ln \frac{v'}{v})'] \tag{4.2}$$

$$v = \frac{1}{\tilde{u}}, \qquad u = \tilde{u}[\tilde{u}\tilde{v} + (\ln \frac{\tilde{u}'}{\tilde{u}})']$$

Having written substitution (4.2) as a result of multi-time application of discrete transformation to a some initial solution we obtain the infinite chain of equations for unknown function $x_n = \ln v_n$

$$x_n'' = x_n'(\exp(x_n - x_{n-1}) - \exp(x_{n+1} - x_n))$$

The reader can find in [4] the explicit solution of this chain under appropriate boundary conditions.

Frechet matrix of substitution is the following

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2(uv) - \frac{v''}{v'} + (1 + \frac{v''v}{v^2}D - \frac{v}{v'}D^2 \end{pmatrix}$$

Keeping in mind the results of the last section one can convinced by direct calculations that the following two operators J_0, J_1

$$J_0(u,v) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J_1(u,v) = \begin{pmatrix} uD^{-1}u' + u'D^{-1}u & D - uv + uD^{-1}v' - u'D^{-1}v \\ D + uv - vD^{-1}u' + v'D^{-1}u & -vD^{-1}v' - v'D^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy or, and this is the same, other solutions of equation (3.1) may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

4.1.3 Nonlinear Schrödinger hierarchy with derivative

$$\dot{u} + u'' - 2(u(uv))' = 0 \quad -\dot{v} + v'' + 2(v(uv))' = 0$$

The direct and inverse substitution in this case have the form

$$\tilde{u} = v, \qquad \tilde{v} = u - (\frac{1}{v})' \tag{4.3}$$

$$v = \tilde{u}, \qquad u = \tilde{v} + (\frac{1}{\tilde{u}})'$$

Having written substitution (4.3) as a result of multi-time application of discrete transformation to a some initial solution we obtain the infinite chain of equations for unknown function $x_n = v_n$

$$x_n' = x_n^2(x_{n+1} - x_{n-1})$$

The reader can find in [4] the explicit solution of this chain Under appropriate boundary conditions.

Frechet matrix of substitution under consideration is the following

$$\phi'(u) = \begin{pmatrix} 0 & 1 \\ 1 & -2\frac{v'}{v^3} + \frac{1}{v^2}D \end{pmatrix}$$

Keeping in mind the results of the last section one can convince by direct calculations that the following two operators J_0, J_1

$$J_1(u,v) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

$$J^0 = \begin{pmatrix} -2uD^{-1}u & 2uD^{-1}v + 1 \\ 2vD^{-1}u - 1 & -2vD^{-1}v \end{pmatrix}$$

satisfy the second equation (3.1).

$F_0 = (u, -v)$ is the simplest nontrivial solution of equation (2.2). All other equations of this hierarchy or, and this is the same, other solutions of equation (3.1) may be obtained by multi-time application of operator $H = J_1 J_0^{-1}$ to this solution.

We represent also the expression for J^2 Hamiltonian operator in illustrative purposes

$$J^2 = \begin{pmatrix} -2u^2D - 2u'u + 2u'D^{-1}u' & D^2 - 2uvD - 2u'v + 2u'D^{-1}v' \\ -D^2 - 2uvD - 2v'u + 2v'D^{-1}u' & -2v^2D - 2vv' + 2v'D^{-1}v' \end{pmatrix}$$

4.1.4 Nonlinear Schrödinger hierarch with derivative and nonlinearity of the third degree

The system of Ablovitz-Kaup

$$-\dot{u} + u'' - 2u^2(v' + uv^2) = 0 \quad \dot{v} + v'' + 2v^2(u' - vu^2) = 0 = 0$$

and Lund-Pohlmeyer-Regge equations

$$\dot{u}' - 4u - 2\dot{u}(uv) = 0 \quad \dot{v}' - 4v + 2\dot{v}(uv) = 0$$

possesses the same discrete transformation and so belong to the single hierarchy.

The direct and inverse discrete substitution in this case are the following

$$\tilde{u} = \frac{1}{v' + uv^2}, \quad \tilde{v} = -(v' + uv^2)' + \frac{(v' + uv^2)^2}{v} \tag{4.4}$$

$$v = -\frac{1}{\tilde{u}' - \tilde{v}(\tilde{u})^2}, \quad u = -(\tilde{u}' - \tilde{v}(\tilde{u})^2)' + \frac{(\tilde{u}' - \tilde{v}(\tilde{u})^2)^2}{\tilde{u}}$$

The discrete transformation in the form of infinite chain in this case coincides with infinite Lotky-Volterra system

$$N_j' = N_i(N_{j+1} - N_{j-1})$$

where $N_{2i} = \frac{v_{2i}}{(v_{2i-1}' + u_{2i-1}v_{2i-1}^2)}$, $N_{2i-1} = -\frac{(v_{2i-1}' + u_{2i-1}v_{2i-1}^2)}{v_{2i}}$

It is not difficult to check that after introduction of the new unknown pair of functions $q = u, r = v' + uv^2$ or $q = u' - vu^2, r = v$ in both cases q, r satisfy the usual nonlinear Schrödinger equation from the first subsection. This fact allows to construct the corresponding Hamiltonian operators using the results of first subsection.

For inverse Frechet operator we obtain immediately

$$(\phi')^{-1} = \begin{pmatrix} -\frac{1}{(\tilde{u}v)^2} & -\frac{2}{\tilde{u}v} - D \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D - 2\tilde{u}\tilde{v} & -(\tilde{u})^2 \end{pmatrix}$$

And after this we obtain for Hamiltonian operators

$$J_{-1} = \begin{pmatrix} 0 & (D - 2uv)^{-1} \\ (D + 2uv)^{-1} & 0 \end{pmatrix} \tag{4.5}$$

$$J_0 = \begin{pmatrix} -2uD^{-1}u & +1 + 2uD^{-1}v \\ -1 + 2vD^{-1}u & -2vD^{-1}v \end{pmatrix} \tag{4.6}$$

4.2 XYZ-hierarchy in classical region.

The system of equations describing the Heisenberg unhomogeneous ferromagnetic in classical region, the equation of Landau-Lifshitz (L-L), is the single known (at least to authors) example of integrable systems with elliptic spectral parameter. We will pay some more attention to this case.

In vector notations L-L system has the form

$$\dot{(\vec{S})} = \vec{S} \times S'' + \vec{S} \times \vec{(JS)}$$

$$\vec{S} = (S_1, S_2, S_3), \quad (\vec{S})^2 = 1, \quad J = diag(J_1, J_2, J_3)$$

Under the steriographic projection

$$u = \frac{S_1 + iS_2}{1 + S_3} \qquad v = \frac{S_1 - iS_2}{1 + S_3}$$

and exchanging $-it \to t$ it became a system of equations:

$$\begin{aligned} -\dot{u} &= u'' - 2v\frac{(u')^2 + R(u)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial u}R(u) \equiv u^{(2)} = 0 \\ \dot{v} &= v'' - 2u\frac{(v')^2 + R(v)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial v}R(v) \equiv v^{(2)} = 0 \end{aligned} \tag{4.7}$$

where $R(x) = \alpha x^4 + \gamma x^2 + \alpha$ $\frac{\partial R}{\partial x} = 4\alpha x^3 + 2\gamma x = 2\frac{R + \alpha(x^4 - 1)}{x}$ $\alpha = \frac{J_2 - J_1}{4}$ $\gamma = \frac{J_1 + J_2}{2} - J_3$ The system (4.7) is invariant under transformation $u \to U, v \to V$:

$$\tilde{u} = \frac{1}{v} \qquad \frac{1}{1 + \tilde{v}\tilde{u}} \quad \frac{1}{1 + uv} = \frac{v\tilde{v}'' - (\tilde{v}')^2 + \alpha(\tilde{v}^4 - 1)}{(\tilde{v}')^2 + R(\tilde{v})} \tag{4.8}$$

which is the discrete substitution for this system. The inverse substitution to (4.7) is the following

$$v = \frac{1}{\tilde{u}} \qquad \frac{1}{1 + vu} - \frac{1}{1 + \tilde{u}\tilde{v}} = \frac{\tilde{u}\tilde{u}'' - (\tilde{u}')^2 + \alpha(\tilde{u}^4 - 1)}{(\tilde{u}')^2 + R(\tilde{u})} \tag{4.9}$$

In the form of infinite chain substitution (4.8) may be rewritten in two equivalent forms

$$\frac{1}{1 + \frac{v_{n+1}}{v_n}} \quad \frac{1}{1 + \frac{v_n}{v_{n-1}}} = \frac{v_n v_n'' - (v_n')^2 + \alpha(v_n^4 - 1)}{(v_n')^2 + R(v_n)}$$

or

$$\frac{1}{v_{n+1}+v_n}+\frac{1}{v_n+v_{n-1}}=\frac{v_n''+R_{v_n}(v_n)}{(v_n')^2+R(v_n)}$$

Reader can find the corresponding solution in [5],[6].

4.2.1 Nondegenerate case

In the case of arbitrary α,γ the main equation (2.3) does not possesses the solution F_0 the components of which have dependence only upon on u,v but not of their derivatives, as solution $F_0=(u,-v)$ of the previous subsections. Due to this reason the second equation (3.1) in the case of XYZ hierarchy possesses the solutions J_n only of the even order but in the degenerated cases solution of this kind takes place. And as corollary in the degenerate cases there take place the Hamiltonian operators of odd and even degree.

We have

$$J_0=(1+uv)^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

The matrix elements of Hamiltonians operators in what follows will be sufficiently complicated to write them in the single matrix form. So we will introduce notations

$$J_s=\begin{pmatrix}a_s&b_s\\-b_s^T&d_s\end{pmatrix}$$

and represent the explicit expressions for a_s,b_s,d_s in all other cases.

So we obtain

$$a_2=2u^{(2)}D^{-1}u'+2u'D^{-1}u^{(2)}$$

where notations $u^{(2)},v^{(2)}$ are introduced in (4.7) and explicit form of which is the following

$$u^{(2)}=u''-\frac{2v(u')^2+2\alpha v+\gamma vu^2-\gamma u-2\alpha u^3}{1+uv}$$

$$v^{(2)}=-v''+\frac{2u(v')^2+2\alpha u+\gamma uv^2-\gamma v-2\alpha v^3}{1+uv}$$

Next

$$b_2=(1+uv)^2D^2+4uv'(1+uv)D+(p'+p^2+2H)(1+uv)^2+2u^{(2)}D^{-1}v'+2u'D^{-1}v^{(2)}$$

where $p = \frac{2uv'}{1+uv}$ the impulse and $H = \frac{2(u'v'+\alpha(u^2+v^2)-\gamma uv)}{(1+uv)^2}$ hamiltonian of L-L equation. And at last

$$d_2 = 2v^{(2)}D^{-1}v' + 2v'D^{-1}v^{(2)}$$

In vector notations the explicit expression for J_2 Hamiltonian operator was obtained firstly in [7].

4.2.2 Degenerate cases

In the partial cases $\alpha = 0, \gamma = \pm 2\alpha$ the symmetry of L-L equation is changed by the jump (it became more wide). The main equation (2.3) possesses solutions which do not depend on derivatives on u, v and as a consequence it is possible to find solution of the equation (3.1) also for odd n.

In the case $\gamma = 2\alpha$ for components of J_1 Hamiltonian operator we obtain

$$a_1 = 2u'D^{-1}u' + 2\alpha(u^2+1)D^{-1}(u^2+1)$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' + 2\alpha(u^2+1)D^{-1}(1+v^2)$$

$$d_1 = 2v'D^{-1}v' + 2\alpha(v^2+1)D^{-1}(v^2+1)$$

Ii is not difficult to check that independent on derivatives solution of (2.3) has the form $F_0 = (u^2+1, v^2+1)$.

In the case $\gamma = -2\alpha$ the corresponding values are the follows

$$a_1 = 2u'D^{-1}u' + 2\alpha(u^2-1)D^{-1}(u^2-1)$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' - 2\alpha(u^2-1)D^{-1}(v^2-1)$$

$$d_1 = 2v'D^{-1}v' + 2\alpha(v^2-1)D^{-1}(v^2-1)$$

$F_0 = (u^2-1, -v^2+1)$.

And at last in the case $\alpha = 0$

$$a_1 = 2u'D^{-1}u' + 2\gamma uD^{-1}u$$

$$b_1 = (1+uv)^2D + 2uv'(1+uv) + 2u'D^{-1}v' - 2\gamma uD^{-1}v$$

$$d_1 = 2v'D^{-1}v' + 2\gamma vD^{-1}v$$

$F_0 = (u, -v)$.

In vector notations this result was obtained in paper [?].

5 Matrix hierarchies

In order to complete our previous results,we are going to present you the generalization for noncommutative variables in $(1+1)$ case. We represent our results in brief form for the hierarchies of Nonlinear Schrödinger and Nonlinear Schrödinger with derivative. In spite of our generalization for the structure of invariant Hamiltonian operators,we were to use the primary scheme (3) in order to find the correct expressions in noncommutative case. Though it is possible to guess such operators.

5.1 Nonlinear matrix Schrödinger hierarchy

$$\dot{u} = -u_{xx} + 2(uvu) \quad -\dot{v} = -v_{xx} + 2(vuv)$$

We consider variables u and v as nonsingular square matrices of N^2 elements. The generalization of the corresponding direct and inverse substitutions takes the form

$$\tilde{u} = v^{-1} \quad \tilde{v} = vuv - v_{xx} + v_x v^{-1} v_x \tag{5.1}$$

$$v = \tilde{u}^{-1} \quad u = \tilde{u}\tilde{v}\tilde{u} - \tilde{u}_{xx} + \tilde{u}_x \tilde{u}^{-1} \tilde{u}_x$$

If we rewrite substitution (5.1) we will obtain the infinite matrix Toda chain of equations for unknown function $w_n = v$

$$(w_{nx} w_n^{-1})_x = w_n w_{n-1}^{-1} - w_{n+1} w_n^{-1}$$

Frechet derivative have the following form

$$\varphi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

α,β,δ,γ are the matrix operators with the following components

$$\alpha_{ij}^{kl} = 0 \quad \beta_{ij}^{kl} = -(v^{-1})_{ki}(v^{-1})_{jl} \quad \gamma_{ij}^{kl} = v_{ki} v_{jl}$$

$$\delta^{kl}_{ij} = \delta_{ki}(uv)_{jl}+(vu)_{ki}\delta_{jl}-(v_xv^{-1})_{ki}(v^{-1}v_x)_{jl}+[\delta_{ki}(v^{-1}v_x)_{jl}+(v_xv^{-1})_{ki}\delta_{jl}]D_x-\delta_{ki}\delta_{jl}D_x^2$$

Each of the indexes k,l,i,j runs from 1 to N.

The Hamiltonian operators for this hierarchy are the following

$$J^0 = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$$

$$T^{kl}_{ij} = \delta_{kj}\delta_{il} \quad k,l,i,j = 1...N$$

$$J^1 = \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix}$$

$$a^{kl}_{ij} = u_{kj}D^{-1}u_{il} + u_{il}D^{-1}u_{kj} \quad b^{kl}_{ij} = \delta_{kj}\delta_{il}D - \delta_{il}u_{ks}D^{-1}v_{sj} - \delta_{kj}u_{sl}D^{-1}v_{is}$$

$$d^{kl}_{ij} = v_{kj}D^{-1}v_{il} + v_{il}D^{-1}v_{kj}$$

k,l,i,j=1...N

One may easily convince by direct calculations that the operators J^0,J^1 satisfy the second equation (3.1)

5.2 Nonlinear matrix Schrödinger hierarchy with derivative

$$\dot{u} = -u_{xx} + 2(uvu)_x \quad \dot{v} = v_{xx} + 2(vuv)_x$$

The generalization of the corresponding direct and inverse substitution (4.3) is the following

$$\tilde{u} = v \quad \tilde{v} = u + v^{-1}v_xv^{-1}$$

(5.2)

$$v = \tilde{u} \quad u = \tilde{v} - \tilde{u}^{-1}\tilde{u}_x\tilde{u}^{-1}$$

If we rewrite the substitution (5.2) we obtain the infinite Toda chain of equations for unknown function $w_n = v$

$$w_{nx} = w_n(w_{n+1} - w_{n-1})w_n$$

Frechet derivative takes the following form

$$\varphi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\alpha_{ij}^{kl} = 0 \quad \beta_{ij}^{kl} = \delta_{ki}\delta_{jl} \quad \gamma_{ij}^{kl} = \delta_{ki}\delta_{jl}$$

$$\delta_{ij}^{kl} = -(v^{-1})_{ki}(v^{-1}v_x v^{-1})_{jl} - (v^{-1}v_x v^{-1})_{ki}(v^{-1})_{jl} + (v^{-1})_{ki}(v^{-1})_{jl}D_x$$

Each of the indexes k,l,i,j runs from 1 to N

Hamiltonian operators J^0 and J^1 for this hierarchy are the following

$$J^0 = \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix}$$

$$a_{ij}^{kl} = -u_{kj}D^{-1}u_{il} - u_{il}D^{-1}u_{kj} \quad b_{ij}^{kl} = \delta_{kj}\delta_{il} + \delta_{il}u_{ks}D^{-1}v_{sj} + \delta_{kj}u_{sl}D^{-1}v_{is}$$

$$d_{ij}^{kl} = -v_{kj}D^{-1}v_{il} - v_{il}D^{-1}v_{kj}$$

k,l,i,j=1...N

$$J^1 = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} D_x$$

$$T_{ij}^{kl} = \delta_{kj}\delta_{il} \quad k,l,i,j = 1...N$$

One may easily verify that the operators J^0 and J^1 satisfy the second equation (3.1)

We see in both cases that the limit $N = 1$ gives us 2 in corresponding operators.

6 Concluding

The main concrete result of the present paper consists of explicit formulas for Hamiltonian operators which allow to reconstruct all equations for most often used hierarchies of integrable systems in physical applications. We have not included in this list results concerning main chiral field problem. The reason of that is the fact of nonlocality of corresponding substitution and

some specific difference of this case in compare with the examples considered in this paper. We hope to return to this problem in recent future.

Methods of construction of Hamitonian operators,which were used in this paper, are typical for group representation theory. The equations (7) from group theoretical point of view are the equations on some invariant kernels. The main equation (2.2) (with given integrable substitution $\varphi(u)$) is th determination of some linear representation of integrable substitution group as was mentioned above. So independent investigation and construction of the representation theory of this object is equivalent to the theory of integrable systems of the given hierarchy. That is the second and may be more important conclusion which follows from the results of this paper.

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References

[1] P.J.Olver Application of Lie Groups to differential equations (Springer, Berlin, 1986).

[2] D.B.Fairlie, A.N.Leznov The Integrable Mapping as the Discrete Group of Internal Symmetry of Integrable Systems Preprint DTP/93/33, Durham (1993).

[3] A.N.Leznov, A.V.Razumov The Canonical Symmetry for Integrable Systems J.Math.Phys.,35,(1994),1738-1754.
A.N.Leznov, A.V.Razumov Hamiltonian Properties of the Canonical Symmetry J.Math.Phys.,35,(1994),4067-4087.

[4] A. N. Leznov, Nonlinear Symmetries of Integrable Systems *J.of Sov.Lazer. Research***3-4**,278-288, (1992)

A.N.Leznov Backlund Transformation for Integrable Systems Preprint IHEP-92-112 DTP,(1992).

[5] B.N.Belov, A.N.Leznov and W.J.Zakrzewski Generalization of the Toda Chain to Elliptic Curve Case Preprint IHEP-94-55 DTP,(1994)

[6] Y.N.Sidorenko., *The questions of quantum field theory and statistic physics*, notes of scientific seminars of LOMI (161), N 7, 76-87, (1987)

E.Barouch,A.S.Fokas and V.G.Papageorgiou *J.Math.Phys* (29), N 12, 2628-2633, (1988).

T.Bemmelen and P.Kersten *J.Math.Phys* (32) N 7, 1709-1716, (1991)

[7] E.V.Ferapontov *Teor.Math.Phys* (91), N 3, 452-463, (1992)

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